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CONTENTS

On the derivates of a function

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Introduction.

1. The most general results as regards the derivates of a continuous function are those given by Prof. Denjoy and Prof. Young. Prof. Denjoy's result* is:

If $f(x)$ be a continuous function, finite at each point and if a set of measure zero be left out of account, then, at the various points x, only the following four cases are possible;

(1) $D^+=D^-=D_+=D_-=$finite,

(2) $D^+=D^-=\infty$; $D_+=D_-=-\infty$,

(3) $D^+=\infty$, $D_-=-\infty$; $D_+=D^-=$finite,

(4) $D^-=\infty$, $D_+=-\infty$; $D_-=D^+=$finite.

The result† given by Prof. Young is the following:

There is no distinction of right and left as regards the derivates of a continuous function except at a set of the first category.

It will be observed that the above theorems give the relations that subsist between the *four derivates* at the various points x, in the interval of definition of $f(x)$, if, as in Denjoy's theorem we leave out of account a set of measure zero, or as in Young's theorem we leave out a set of the first category.

In the present paper, instead of taking the four derivates together, we consider (1) the relations that subsist between the two derivates, on the same side, at the various points x, in the interval of definition of

* Denjoy : Memoire sur les nombres derives des fonctions continue. *Journ. de Math.*, (7), Vol. I (1915), pp. 105-240.

† W. H. Young : Oscillating successions of continuous functions. *Proc. London Math. Soc.* (2). Vol. 6, pp. 298-320; also see, 'On derivates and the theorem of the mean', *Quart. Journ. of Math.*, Vol. 40 (1909), pp. 1-26.

$f(x)$; and (2) the values that one of the derivates (say D^+) can have at the various points x, in the interval of definition of $f(x)$.

Denjoy's theorem states that the two derivates on the same side cannot both be *finite* and different except at a set of measure zero. The question naturally arises: What relations, if any, subsist between the two derivates on the same side if we do not restrict them to be finite? An answer to this question is given by theorem I of this paper.

We know that one of the derivates (say D^+) can be finite everywhere. Theorem II of this paper shows that the upper (lower) derivate cannot be $+\infty$ $(-\infty)$ at all the points of the interval of definition of $f(x)$, with the possible exception of those that belong to an enumerable set, which may be everywhere dense, and those that belong to a non-dense set which may be of positive measure.

A number of interesting corollaries to the above theorems have also been given. It is not kn wn whether the exceptional set of measure zero, which occurs in the enunciation of Denjoy's theorem, always exists.* By the help of the results obtained in this paper it has been found possible to enumerate the cases in which the exceptional set must exist.

All the results of this paper are believed to be new.

2. THEOREM I. *If $f(x)$ be finite and continuous or quasi-continuous*† *in the interval (a, b), then, it cannot possess a right (left) hand upper derivate that is greater than a finite number σ_1, and a right (left) hand lower derivate that is less than another finite number σ_2, $(\sigma_2 < \sigma_1)$, at each point of the interval (a, b), which does not belong to an enumerable set G and a non-dense set N, at which nothing is known as regards the values of the derivates.*

Proof: Let $f(x)$ be a finite and continuous or quasi-continuous function in the interval (a, b), and, if possible, let

$$D^+f(x) > \sigma_1 \text{ and } D_+f(x) < \sigma_2,$$

at all the points x in (a, b) which do not belong to an enumerable set G and a non-dense set N; where σ_1 and σ_2 are finite numbers such that $\sigma_2 < \sigma_1$.

* As regards the exceptional set of the first category which occurs in the enunciation of Young's theorem, it is known that it exists in the case of all non-differentiable functions. See W. H. Young, On the derivates of non-differentiable functions, *Messenger of Mathematics*, Vol. 38, (1908-9), pp. 65-69.

† A function is said to be quasi-continuous if its points of discontinuity form a non-dense set. At the points of discontinuity the function may be infinite.

As $f(x)$ is quasi-continuous in (a, b), it is possible to find an interval (a', b'), in (a, b), in which the function is continuous. Further, as the set N is non-dense, it is possible to find an interval (α, β), in (a', b'), in which there is no point of N. Let G′ be the part of G contained in the interval (α, β). Then as G′ is enumerable, its points can be exhibited as a series

(2.1) $$u_1, u_2, u_3, \ldots\ldots u_n, \ldots\ldots$$

Corresponding to the point u_n $(n=1,2,\ldots)$, let there be assigned the greatest interval δ_n, with u_n as left end point, such that

(2.2) $$\delta_n \leq \frac{\epsilon}{2^n},$$

and

(2.3) $$| f(u_n) - f(x) | \leq \frac{w}{2^n},$$

if $$| u_n - x | \leq \delta_n,$$

where ϵ and ω are arbitrarily chosen small positive numbers. Such intervals exist because the function is continuous in (α, β).

We can now construct a chain of intervals * in (α, β) as follows:

If α does not belong to the exceptional set G′, then, with α as left end point we find the greatest interval (α, x_1), such that

$$\frac{f(x_1) - f(\alpha)}{x_1 - \alpha} \geq \sigma_1.$$

Such an interval exists for $D^+ f(\alpha) > \sigma_1$.

If α belongs to G′, it is a point u_r of the series (2.1), and we assign to α, the interval

$$\delta_r \equiv (\alpha, x_1).$$

Again, to x_1 let there correspond the greatest interval (x_1, x_2), such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq \sigma_1,$$

or the interval

$$\delta_q = (x_1, x_2)$$

according as x_1 does not belong to G′, or is the point u_q of G′.

* The following portion of the proof is greatly simplified if we assume the existence of the chain of intervals.

Constructing as above we get a chain of intervals

$$(a, x_1), (x_1, x_2), (x_2, x_3), \ldots\ldots$$

This chain may or may not reach β. If it does not reach β, the end points of the intervals of the chain have a limiting point ξ within (α, β). We then begin with ξ and construct a chain of intervals as before. If this too, does not reach β, there is as before a limiting point η. We then begin with η and construct a chain of intervals as before.

We repeat the procedure sketched above till we reach p or a point p which lies in the interval $(\beta-\theta, \beta)$, where θ is a predetermined small positive number, such that

$$(2.4) \qquad |f(\beta)-f(x)| \leq \lambda,$$

$$(2.5) \qquad \text{if} \quad |\beta-x| \leq \theta,$$

where λ is an arbitrarily chosen small positive number.

A chain of intervals reaching from α to a point arbitrarily near p exists, for if the process of construction does not come to an end, the cardinal number of intervals required to be constructed within (α, p) would be unenumerable, which is in contradiction to the well known result that every set of non-overlapping intervals is enumerable.

The chain of intervals

$$(2.6) \qquad (\alpha, x_1), (x_1, x_2), \ldots\ldots\ldots(\xi, \xi_1), (\xi_1, \xi_2)\ldots$$
$$\ldots\ldots\ldots(\eta, \eta_1), (\eta_1, \eta_2)\ldots\ldots(p, \beta)$$

reaching from α to β, is composed of

(2.7) a set of intervals with points of G' as left end point, which we denote by $\{\Delta_r\}$,

(2.8) a set of intervals whose left end points do not belong to G', and such that the incrementary ratio of the end points of each interval of the set is greater than or equal to σ_1, which we denote by $\{S_r\}$;

(2.9) and the interval* (p, β) such that

$$|f(p)-f(\beta)| \leq \lambda.$$

Now

$$(2.10) \qquad f(\beta)-f(\alpha)=\{f(x_1)-f(\alpha)\}+\{f(x_2)-f(x_1)\}$$
$$+\ldots\ldots+\{f(\xi_1)-f(\xi)\}+f(\xi_2)-f(\xi_1)\}$$
$$+\ldots\ldots+\{f(\eta_1)-f(\eta)\}+\{f(\eta_2)-f(\eta_1)\}$$
$$+\ldots\ldots+\{f(\beta)-f(p)\}$$

* This is absent, in case the chain of intervals reaches β.

The sum of the function differences for the intervals $\{\Delta_r\}$ is

$$(2.11) \qquad \leq \sum_1^\infty \frac{w}{2^n} \text{ i.e., } \leq \omega \text{ by } (2.3).$$

and the sum of their lengths is

$$(2.12) \qquad \leq \sum_1^\infty \frac{\epsilon}{2^n} \text{ i.e., } \leq \epsilon \text{ by } (2.2).$$

Therefore, the sum of the lengths of the intervals $\{S_r\}$

$$(2.13) \qquad \sum S_r \text{ is } \geq (\beta-\alpha)-\epsilon-(\beta-p)$$
$$\text{i.e., } \geq (\beta-\alpha)-\epsilon-\theta.$$

Hence

$$(2.14) \qquad f(\beta)-f(\alpha) \geq \sigma_1 \sum S_r - \omega - \{f(\beta)-f(\alpha)\}$$
$$\geq \sigma_1\{(\beta-\alpha)-\epsilon-\theta\}-\omega-\lambda$$

But ϵ, θ, ω and λ can be each taken as small as we please,

$$(2.15) \qquad \therefore \; f(\beta)-f(\alpha) \geq \sigma_1(\beta-\alpha)-k, \text{ where } k \text{ is arbitrarily small.}$$

In a like manner, as $D_+f(x) < \sigma_2$, we can by similarly constructing a chain of intervals show that

$$(2.16) \qquad f(\beta)-f(\alpha) \leq \sigma_2(\beta-\alpha)+k.$$

As k is arbitrary, and $\sigma_2<\sigma_1$, the results (2.15) and (2.16) are contradictory. Therefore, our supposition is untenable and theorem I. holds.

3. *Corollary I. If $f(x)$ be finite and continuous or quasi-continuous in (a,b), then, it cannot possess at each point in (a, b), with the possible exception of the points of an enumerable set and a non-dense set, a right (left) hand upper derivate which is equal to ∞ and a right (left) hand lower derivate which is equal to $-\infty$.*

This follows at once from Theorem 1. We see that the possibility (2) of Denjoy's theorem cannot occur everywhere, so that, in this case, an exceptional set exists. When $f(x)$ is not totally non-differentiable it can be easily shown by means of a suitably constructed example that the exceptional set may be everywhere dense and of positive measure. When $f(x)$ is non-differentiable, it follows from a result of W. H. Young * that the exceptional set exists and is of the first category.

* W. H. Young : On the derivates of non-differentiable functions, *Messenger of Mathematics*, Vol. 38 (1908-9), pp. 55-69.

Corollary I further enables us to state that the exceptional set of the first category must be such that it is *not* composed of an enumerable set and a non-dense set, in other words, the exceptional set must be unenumerable in every interval inside the interval of definition of $f(x)$.

Corollary II. Associated with every finite and quasi-continuous function $f(x)$, defined in the interval (a,b), is an unenumerable every where dense set of points in (a,b), at no point of which, the inequalities*

$$D^+f(x) > \sigma_1,\ D_+f(x) < \sigma_2,$$

are together satisfied; σ_1 and σ_2 being any two finite numbers such that $\sigma_2 < \sigma_1$.

Corollary III. If $f(x)$ is a finite and quasi-continuous function in the interval (a,b), then, there exists an unenumerable everywhere dense set of points in (a,b), at each point of which one of the following two cases occurs:

$$(a) \quad \alpha < D_+ \leq D^+,$$

$$(b) \quad D_+ \leq D^+ < \beta,$$

where α and β are any two arbitrarily assigned numbers, such that $\alpha < \beta$.

The above result can be easily deduced from theorem I, and is more general than the following known result: †

If E be a set of points in the interval (a,b), at which $Df(x)$, one of the derivates of a function $f(x)$, continuous in (a,b), has a fixed sign (and is not zero), the set E, when it exists, is unenumerable, and contains a perfect set.

4. THEOREM II. *A finite and quasi-continuous function $f(x)$, defined in the interval (a,b), cannot possess an upper right (left) hand derivate which is equal to ∞ at each point of (a,b) which does not belong to an enumerable set G and a non-dense set N.*

Proof: Let $f(x)$ be finite and quasi-continuous in the interval (a,b), and if possible let $D^+f(x) = \infty$ at each point x in (a,b) which does not belong to an enumerable set G and a non-dense set N.

* The set is such that it is unenumerable in every interval in (a,b).

† See de la Valee Poussin's *Cours d'Analyse*, 2nd. ed., Vol. I, p. 80. The proof given there is incorrect as has been pointed out by Hobson. *Cf.* Hobson's *Theory of Functions*, 3rd. ed., Vol. I, p. 586.

As in the proof of theorem I, we can find an interval (α,β) which does not contain a point of N and in which $f(x)$ is finite and continuous. Thus $f(\beta)-f(\alpha)$ is a finite number. Let G′ be the component of G in (α,β).

Let M be an arbitrarily chosen large positive number.

Then with a point x, which does not belong to G′ as left end point, there exists a greatest interval $\delta_x \equiv (x, x+h)$, the incrementary ratio of whose end points.

$$\frac{f(x+h)-f(x)}{h} \geq \mathrm{M} \tag{4.1}$$

Further, with each point of G′ as left end point, we can assign a definite interval as in (2.1), (2.2) and (2.3).

Now, as in the proof of the previous theorem, we can

(A) construct a chain of intervals reaching from α to β, such that there is a set of intervals belonging to this chain, the sum of whose lengths is

$$> (b-a)-\epsilon, \tag{4.2}$$

where ϵ is arbitrarily small; and for each of which the incrementary ratios formed by the end points is

$$\geq \mathrm{M}, \tag{4.3}$$

while the sum of the absolute values of the function-differences for the remaining intervals of the chain is less than or equal to an arbitrarily assigned small positive number η and thus

(B) show that

$$f(\beta)-f(\alpha) > \mathrm{M}(\beta-\alpha)-\kappa \tag{4.4}$$

where κ is arbitrarily small.

The function $f(x)$ being finite, $f(\beta)-f(\alpha)$ is fixed and finite so that (4·4) is absurd, and therefore, theorem II holds. It follows that *The set of points where $D^+f(x)$ is not equal to ∞ must be unenumerable and such that it is not composed of an enumerable set and a non-dense set.*

The above reasoning holds for any interval (p,q) in (a,b); therefore, the points where $D^+f(x)$ is not equal to ∞ must form an everywhere dense set in (a,b). We have, therefore, the following result;

If $f(x)$ be finite and quasi-continuous, in the interval (a,b), there exists an unenumerable everywhere dense set of points in (a,b) at each point of which $D^+f(x)$ is not equal to ∞.

Similar results hold for the lower derivate.

Corollary IV. A finite and quasi-continuous function defined in the interval (a,b), cannot possess a right (left) hand lower derivative that is equal to $-\infty$ at each point of (a,b) which does not belong to an enumerable set G and a non-dense set N. In other words, there exists an unenumerable everywhere dense set of points E, in (a,b), at each point of which the lower derivate on the right (left) is not $-\infty$. The set E is such that the part of it contained in any interval in (a,b) is unenumerable.

Corollary V. The following well-known theorem follows as a corollary: *a continuous functions $f(x)$ cannot have, at every point of a whole interval, a single valued derivative on the right (left) which is everywhere infinite and of the same sign.**

Corollary VI. The cases (2) and (3) or (2) and (4) of Denjoy's theorem cannot occur everywhere in a whole interval, with the possible exception of an enumerable set and a non-dense set.

The cases (2) and (3) of Denjoy's theorem are:

(2) $$D^+=D^-=\infty\,;\ D_+=D_-=-\infty$$

(3) $$D^+=\infty,\ D_-=-\infty\,;\ D_+=D^-=\text{finite.}$$

If these occur, then, $D^+=\infty$ at each point of (a,b) which does not belong to an enumerable set and a non-dense set, which by theorem II is impossible, if $f(x)$ be finite and continuous or quasi-continuous.

The second part of the corollary can be similarly shown to hold.

It follows from our discussion that out of the four cases enumerated in Denjoy's theorem, only the case (1) or the cases (2), (3) and (4) together may occur everywhere in an interval.

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* This follows as a particular case of theorem II. It has been proved in Hobson's *Theory of Functions*, etc., 3rd. ed., Vol. I, p. 385; also see Dini: *Fundamenta per la Teorica della Funzioni di Variabili Reali*, p. 177.

2

On a Substitution and the Equivalence of Two Forms

By

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Introduction.

It is a well-known proposition[1] that if S denotes any uni-proper substitution[2] which converts a pro-primitive form[3] (a, b, c) of determinant $D = b^2 - ac$ into another (a', b', c'), then all such substitutions are given by $R^i S$, $(i = \pm 1, \pm 2, \ldots\ldots)$, where

$$(1) \qquad \ldots \qquad R = \begin{pmatrix} T - bU, & -cU \\ aU, & T + bU \end{pmatrix},$$

(T, U) being the *fundamental solution, i.e.*, a solution for which T and U have the least positive integral values, of the equation

$$(2) \qquad \ldots \qquad x^2 - Dy^2 = 1$$

Now a question arises as to how to obtain the substitution S ; if the coefficients a, b, c and a', b', c' are numerical, then the well-known *classical method* [4] whereby a *reduced* form equivalent to each of (a, b, c) and

[1] G. B., Mathews' *Theory of Numbers*, Part I, 1892, Art 89; hereafter this book will be referred to as Mathews' *Numbers*.

[2] A substitution $S = \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$ will be called *uni-proper* or *unim-proper* according as $\alpha\delta - \beta\gamma = +1$ or $= -1$, it being understood that $\alpha, \beta, \gamma, \delta$ are integers positive or negative.

[3] A form (a, b, c) will be said to be *pro-primitive* if $dv(a, 2b, c) = +1$, the notation $dv(x, y, z, \ldots\ldots)$ being employed to mean according to Prof. Mathews the greatest common divisor of $x, y, z, \ldots\ldots$ taken positively.

[4] *Cf.* Mathews' *Numbers* I, Arts. 66, 80.

(a', b', c') is first found, enables us to determine S but we have not been given a general expression for S from which can be calculated all cases that may arise. The present writer to the best of his knowledge and information is not aware whether such an expression has been found. The principal aim in the following pages is to obtain such a substitution S and then to examine its sphere of applicability to the wholly allied question of the equivalence of two forms with reference to its usefulness and advantages in this direction.

1. If the pro-primitive form (a, b, c) of determinant D is converted into another (a', b', c') by the uniproper sustitution $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$, then will

$$(3) \quad \ldots \quad \alpha = \frac{t-bu}{a}, \quad \beta = \frac{(b'-b)t+(\mathrm{D}-bb')u}{aa'}, \quad \gamma = u, \quad \delta = \frac{t+b'u}{a'}.$$

where (t, u) is such an integral solution of the equation

$$(4) \quad \ldots \qquad x^2 - \mathrm{D}y^2 = aa',$$

as will make each of

$$(5) \quad \ldots \qquad \frac{t-bu}{a}, \qquad \frac{(b'-b)t+(\mathrm{D}-bb')u}{aa'}, \qquad \frac{t+b'u}{a'}$$

an integer.

We have

$$\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix} (a, b, c) = (a', b', c'),$$

so that

$$(6) \quad \ldots \qquad a\alpha^2 + 2b\alpha\gamma + c\gamma^2 = a',$$

$$(7) \quad \ldots \qquad (a\alpha + b\gamma)\beta + (b\alpha + c\gamma)\delta = b,$$

and also because the substitution $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is uniproper,

$$(8) \quad \ldots \qquad \alpha\delta - \beta\gamma = 1.$$

From (6) we get

$$(a\alpha + b\gamma)^2 - \mathrm{D}\gamma^2 = aa'.$$

Now let us make

(9) ... $$t=a\alpha+b\gamma, \qquad u=\gamma,$$

then t and u are integers and we get

(10) ... $$t^2-Du^2=aa'$$

so that (t, u) is an integral solution of the equation (4).

From (9) we get

(11) ... $$\alpha=\frac{t-bu}{a},$$

and now substituting in (7) and (8) the values of $a\alpha+b\gamma$, α and γ in terms of t and u as given in (9) and (11) we get respectively

$$t\beta+\left(b.\frac{t-bu}{a}+cu\right)\delta=b',$$

$$\frac{t-bu}{a}\delta-u\beta=1.$$

Multiplying the first of these by u and the second by t and adding, we get after simplification

$$t+b'u=\frac{\delta}{a}\{b(t-bu)u+acu^2+t(t-bu)\}$$

$$=\frac{\delta}{a}(t^2-Du^2), \qquad \because \quad D=b^2-ac$$

$$=\delta a'. \qquad \text{by (10)}$$

(12) ... $$\therefore\ \delta=\frac{t+b'u}{a'}$$

Finally, from (8), we get

$$\beta=\frac{a\delta-1}{\gamma}=\frac{a\delta-1}{u} \qquad \because \quad \gamma=u \quad \text{by (9)}$$

Substituting in this relation the values of α and δ as obtained in (11) and (12) and simplifying we get

$$\beta=\frac{t^2+(b'-b)tu-bb'u^2-aa'}{aa'u}$$

$$=\frac{Du^2+(b'-b)tu-bb'u^2}{aa'u}, \qquad \because \quad t^2=Du^2+aa' \quad \text{by} \quad (10)$$

We thus get

$$(13) \quad \ldots \qquad \beta = \frac{(b'-b)t + (D-bb')u}{aa'}$$

Hence the substitution $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$ becomes

$$(14) \quad \ldots \qquad S = \begin{pmatrix} \dfrac{t-bu}{a}, & \dfrac{(b'-b)t+(D-bb')u}{aa'} \\ u & \dfrac{t+b'u}{a'} \end{pmatrix}$$

Also since α, δ, β are integers, it follows from (11), (12) and (13) respectively that (t, u) is to be such an integral solution of the equation (4) as will make each of the numbers in (5) an integer.

We now state and prove the converse proposition.

If (t, u) is an integral solution of the equation

$$(4) \quad \ldots \qquad x^2 - Dy^2 = aa',$$

making each of the numbers

$$(5) \quad \ldots \qquad \frac{t-bu}{a}, \qquad \frac{(b'-b)t+(D-bb')u}{aa'}, \qquad \frac{t+b'u}{a'}$$

an integer, then will the sustitution

$$(14) \quad \ldots \qquad S = \begin{pmatrix} \dfrac{t-bu}{a}, & \dfrac{(b'-b)t+(D-bb')u}{aa'} \\ u & \dfrac{t+b'u}{a'} \end{pmatrix}$$

be uniproper and convert the form (a, b, c) into (a', b', c') so that they will be properly equivalent.

It is easily verified with the help of (10) that the substitution (14) is uniproper. To prove that

$$S.(a, b, c) = (a', b', c'),$$

it is found that the coefficient of x'^2 on the right side

$$= a\left(\frac{t-bu}{a}\right)^2 + 2b.\,\frac{t-bu}{a}\,u + cu^2$$

$$=\frac{1}{a}(t^2-2btu+b^2u^2+2btu-2b^2u^2+cau^2)$$

$$=\frac{1}{a}(t^2-Du^2), \quad \because \quad D=b^2-ac$$

$$=a', \text{ by } (10)$$

Next, the coefficient of $2x'y'$

$$=\frac{1}{aa'}\cdot(t-bu)\{(b'-b)t+(D-bb')u\}+\frac{b}{aa'}[(t-bu)(t+b'u)$$

$$+\{(b'-b)t+(D-bb')u\}u]+\frac{c}{a'}(t+b'u)u$$

$$=\frac{1}{aa'}[(t-bu)\{(b'-b)t+(D-bb')u\}+b(t-bu)(t+b'u)$$

$$+bu\{(b'-b)t+(D-bb')u\}+ca(t+b'u)u]$$

$$=\frac{1}{aa'}\{(b'-b+b)t^2+(D-bb'-bb'+b^2+bb'-b^2+bb'-b^2+ca)tu$$

$$+(-bD+b^2b'-b^2b'+bD-b^2b'+b'ca)u^2\}$$

$$=\frac{1}{aa'}\{b't^2-b'(b^2-ac)u^2\}$$

$$=\frac{b'}{aa'}(t^2-Du^2)=b' \quad \text{by (10).}$$

Lastly, the coefficient of y'^2

$$a\left\{\frac{(b'-b)t+(D-bb')u}{aa'}\right\}^2+2b.\frac{(b'-b)t+(D-bb')u}{aa'}.\frac{t+b'u}{a'}$$

$$+c.\left(\frac{t+b'u}{a}\right)^2$$

$$=\frac{1}{aa'^2}[\{(b'-b)^2+2b(b'-b)+ca\}t^2+2\{(b'-b)(D-bb')+bb'(b'-b)$$

$$+b(D-bb')+cab'\}tu+\{(D-bb')^2+2bb'(D-bb')+cab'^2\}u^2]$$

$$=\frac{1}{aa'^2}[(b'^2-2b'b+2bb'-2b^2+ac)t^2+2(b'D-b'^2b-bD+b^2b'$$

$$+bb'^2-b^2b'+bD-b^2b'+cab')tu+(D^2-2bb'D+b^2b'+2bb'D$$
$$-2b^2b'^2+cab'^2)u^2]$$

$$=\frac{(b'^2-D)t^2+\{D^2-b'^2(b^2-ac)\}u^2}{aa'^2}$$

$$=\frac{a'c'(t^2-Du^2)}{aa'^2} \qquad \because \quad D=b^2-ac=b'^2-a'c',$$

$$=c' \quad \text{by (10).}$$

We thus get what is required. It has thus been proved that

The necessary and sufficient condition that the pro-primitive forms (a, b, c) *and* (a', b', c') *of the same determinant* D *may be equivalent is that the equation*

$$(4) \quad \ldots \qquad x^2-Dy^2=aa'$$

possesses an integral solution (t, u) *which makes each of the coefficients*

$$(5) \quad \ldots \qquad \frac{t-bu}{a}, \qquad \frac{(b'-b)t+(D-bb')u}{aa'}, \qquad \frac{t+b'u}{a'}$$

an integer ; when this is the case the uniproper substitution which converts (a, b, c) *into* (a', b', c') *is*

$$(14) \quad \ldots \qquad \begin{pmatrix} \dfrac{t-bu}{a}, & \dfrac{(b'-b)t+(D-bb')u}{aa'} \\ u, & \dfrac{t+b'u}{a'} \end{pmatrix}$$

2. If the two forms (a, b, c) and (a', b', c') become identical so that

$$a=a', \; b=b', \; c=c',$$

then we get

$$\alpha=\frac{t-bu}{a}, \quad \beta=\frac{-cu}{a}, \quad \delta=\frac{t+bu}{a},$$

and the equation (4) changes into

$$(15) \quad \ldots \qquad x^2-Dy^2=a^2,$$

so that the theorem just now stated reduces to the following :—

The necessary and sufficient condition that the substitution

$$(16) \quad \ldots \qquad \begin{pmatrix} \dfrac{t-bu}{a}, & \dfrac{-cu}{a} \\ u, & \dfrac{t+bu}{a} \end{pmatrix}$$

may be an automorph of the pro-primitive form (a, b, c) is that the equation

(15) ... $$x^2-Dy^2=a^2$$

possesses an integral solution (t, u) which renders each of the numbers

(17) ... $$\frac{t-bu}{a}, \quad \frac{-cu}{a}, \quad \frac{t+bu}{a}$$

an integer.

But we know that an automorph of the pro-primitive form (a, b, c) is

(18) ... $$\begin{pmatrix} t'-bu', & -cu' \\ au', & t'+bu' \end{pmatrix},$$

where (t', u') is an integral solution of the equation

(19) ... $$x^2-Dy^2=1.$$

Hence it is necessary to show that conditions of the theorem just now enunciated ensure that the substitution (16) is reducible to that in (18). For this purpose it is only necessary to prove that the necessary and sufficient condition that

(17) ... $$\frac{t-bu}{a}, \quad \frac{-cu}{a}, \quad \frac{t+bu}{a},$$

where (t, u) is an integral solution of the equation

(15) ... $$x^2-Dy^2=a^2$$

may be all integers, is that

(20) ... $$t\equiv u\equiv 0 \pmod{a}$$

If the relation (20) holds good then the numbers in (17) are obviously integers. Conversely, we shall suppose them to be integers and thereby prove the truth of (20).

Let

$$dv(u, a)=\lambda,$$

then we have

$$u=\lambda u', \; a=\lambda a',$$

where λ is an integer and u' and a' are prime to each other.

Now

$$-\frac{cu}{a}=-\frac{c\lambda u'}{\lambda a'}=-\frac{cu'}{a'},$$

and $\because$ $\frac{-cu}{a}$ is an integer by hyp. and u' is prime to a' we get

$$c\equiv 0 \pmod{a'}.$$

Again because $\frac{t-bu}{a}$ and $\frac{t+bu}{a}$ are both integers by hyp. we get

$$\frac{t+bu}{a}-\frac{t-bu}{a}=\frac{2bu}{a}=\frac{2bu'}{a'}=\text{an integer};$$

and as before,

$$2b\equiv 0 \pmod{a'}$$

Thus

$$2b\equiv c\equiv 0 \pmod{a'}$$

$$\therefore \quad dv(a', 2b, c)=a'$$

but the form (a, b, c) being pro-primitive, we have

$$dv(a, 2b, c)=1$$

$$\therefore a'=1,$$

whence we get

$$a=\lambda a'=\lambda, \quad u=\lambda u'=au'.$$

$$\therefore \quad u\equiv 0 \text{ mod } a) \;;$$

and also (t, u) being a solution of the equation (15), we get

$$t^2-\mathrm{D}u^2=a^2,$$

so that

$$t^2\equiv 0 \pmod{a^2}$$

$$\text{or } t\equiv 0 \pmod{a}.$$

We thus get the relation (20). In other words, it is only when the relation (20) is true that the numbers in (17) are integers. It follows, therefore, that we are entitled to put

$$t=t'a, \; u=u'a,$$

where t' and u' are integers and then it follows from (15) that (t', u') becomes an integral solution of the equation (19). The substitution (16) then reduces to

$$\begin{pmatrix} t'-bu' & -cu' \\ au' & t'+bu' \end{pmatrix}$$

which is the same as (18).

We have thus completely investigated the circumstances under which the numbers (17) can be integral but the possibility should not be overlooked that the first and the third of them can be integral

without the second being so and we are thus led to the following theorem:—

If an integral solution (t, u) *for which* $u \not\equiv 0$ *(mod* a*) of the equation*

(15) $$x^2 - Dy^2 = a^2$$

makes both $\dfrac{t-bu}{a}$ *and* $\dfrac{t+bu}{a}$ *integral, then will*

$$u^2 \equiv 0 \quad (\text{mod } a),$$

and a *contain a square factor which also divides* $4D$*, where* D *denotes the determinant of the form* (a, b, c).

Returning to the relations

$$u = \lambda u',\ a = \lambda a',$$

we see that $a' \neq 1$ for otherwise we shall have

$$u = au' \equiv 0 \quad (\text{mod } a)$$

which is contrary to hyp. Also from these relations we get

$$\frac{2bu'}{a'} = \frac{2bu}{a} = \frac{t+bu}{a} - \frac{t-bu}{a} = \text{an integer}$$

and

$$\frac{c\lambda u'^2}{a'} = \frac{cu^2}{a} = 1 - \frac{a^2 - acu^2}{a^2} = 1 - \frac{t-bu}{a} \cdot \frac{t+bu}{a} = \text{an integer,}$$

by hyp.,

and since u' is relatively prime to a', we get

(21) $$2b \equiv 0,\ c\lambda \equiv 0 \quad (\text{mod } a'),$$

Again, since the form (a, b, c) is pro-primitive, we have

$$dv(a, 2b, c) = 1$$

$$\therefore\ dv(a', 2b, c) = 1,\ \because a = \lambda a',$$

and because $2b \equiv 0$ (mod a') by (21), we get c to be prime to a' so that by the second relation in (21) we get

$$\lambda \equiv 0 \ (\text{mod } a')$$

$$\therefore\ a = \lambda a' \equiv 0 \ (\text{mod } a'^2),$$

and as $a' \neq 1$, we conclude that a contains a square factor other than unity. We have then

$$4b^2 \equiv 0,\ 4ac \equiv 0 \ (\text{mod } a'^2)$$

$$\therefore\ 4D = 4(b^2 - ac) \equiv 0 \ (\text{mod } a'^2)$$

Thus the square factor a'^2 divides both a and 4D. Moreover,

$$a'^2 u^2 = a^2 u'^2$$

$$\therefore \quad a'^2 \cdot \frac{u^2}{a} = au'^2 \equiv 0 \pmod{a'^2} \quad \because \quad a \equiv 0 \pmod{a'^2}$$

$$\therefore \quad \frac{u^2}{a} \text{ is an integer}$$

$$i.e., \; u^2 \equiv 0 \pmod{a}$$

Thus the proposition is proved.

Ex. Consider the form $(18, 3, -1)$ of determinant $D=27$ and the equation $x^2 - 27y^2 = 18^2$ of which $(36, 6)$ is an integral solution and $6 \not\equiv 0 (\text{mod. } 18)$. We have also

$$\frac{t-bu}{a} = \frac{36-3\cdot 6}{18} = 1 \text{ and } \frac{t+bu}{a} = \frac{36+3\cdot 6}{18} = 3$$

so that both are integers and consequently we get as we should by the theorem that 18 contains a square factor, *viz.*, 9 and it divides $4\cdot 27$. Also $6^2 \equiv 0$ (mod. 18). It may also be noted that because $u=6 \not\equiv 0$ (mod. 18), $\frac{-cu}{a} = \frac{1\cdot 6}{18}$ is not an integer and accordingly this solution $(36, 6)$ does not lead to an automorph of the form $(18, 3, -1)$.

3. We shall now apply the substitution (14) to examine the equivalence or otherwise of two given forms (a, b, c) and (a', b', c') of the same determinant D and we have here to make out two cases, first, when D is *negative* and secondly, when D is *positive*.

In the first place, suppose D to be negative $= -\Delta$, where Δ is positive, then the equation (4) reduces to

$$x^2 + \Delta y^2 = aa'. \tag{22}$$

The determinant D being negative we can take a and a' to have the same sign which we shall suppose to be positive. We can now at once ascertain whether or no the equation (22) is solvable in integers and as it then possesses a finite number of solutions, we can easily discover such of them, if there be any, as will make each of the numbers (5) an integer. If either the equation (22) is not solvable in integers or when it is so, none of the solutions make the numbers (5) integers, then the forms (a, b, c) and (a', b', c') are not equivalent. But if there is an integral solution which renders the numbers (5) integral, then the forms

(a, b, c) and (a', b', c') ere equivalent and substituting the values of (t, u) so obtained in (14) we get the uniproper substitution which converts (a, b, c) into (a', b', c').

Ex. Examine for equivalence the forms $(9, -8, 11)$, $(79, -132, 221)$ of the same determinant $D=-35$. The equation $x^2+35y^2=711$ has the solution $t=\pm 26$, $u=\pm 1$ and we readily find that $(26, -1)$ make the numbers (5) respectively equal to $2, -3, 2$ which are all integers so that the forms $(9, -8, 11)$, $(79, -132, 221)$ are equivalent and $\begin{pmatrix} 2, & -3 \\ -1, & 2 \end{pmatrix}(9, -8, 11) = (79, -132, 221)$.

This example illustrates how directly and very easily we can examine the equivalence of two given forms without having to pass through a system of reduced forms and at the same time determine the uniproper substitution which converts one into the other without having to compound substitutions in succession. The *classical* method referred to in the Introduction settles the equivalence of the above two forms as follows:—

$$(9, -8, 11) \backsim (11, -3, 4) \backsim (4, -1, 9),$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}(9, -8, 11) = (4, -1, 9);$$

$$(79, -132, 221) \backsim (221, -89, 36) \backsim (36, 17, 9) \backsim (9, 1, 4) \backsim (4, -1, 9)$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}(79, -132, 221) = (4, -1, 9)$$

Hence we get

$$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}(9, -8, 11) = \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}(79, -132, 221)$$

$$\text{or} \quad \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}(9, -8, 11) = (79 -132, 221)$$

$$\text{or} \quad \begin{pmatrix} -2 & 3 \\ 1 & -2 \end{pmatrix}(9, -8, 11) = (79, -132, 221).$$

We thus arrive at the same result as we did by the former method for the substitutions $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$ and $\begin{pmatrix} -2 & 3 \\ 1 & -5 \end{pmatrix}$ are one and the same.

It will be noticed that in each case we arrive at the same reduced form (4,—1, 9) by having to pass through 2 forms in the case of (9,—8, 11) and 4 forms in the case of (79, —132, 221). Moreover, we have to compound 2 substitutions in the one case and 4 in the other and the composition has to be effected by taking two of them only at a time. The number of forms to pass through to reach a reduced form equivalent to a given form varies with the form taken and consequently when it is large the labour and trouble to determine the intermediary *adjacent* forms, the substitutions and their compound for each of the two given forms, rapidly enhance. On the other hand, the method explained in the preceding pages requires us to solve one equation having only a finite number of solutions and to examine the integrality of 3 numbers, whatever be the forms chosen and except on occasions which are not frequent, the equation possesses only four solutions such as $(\pm t, \pm u)$ one of which can readily be found, if the forms are equivalent to make all the three numbers in (5) integral.

To disprove the equivalence of two forms our method is equally potent. It will then happen that either the equation (22) does not possess integral solutions or if it does, none of them will make all the numbers (5) integral. For the same purpose, the *classical* method of *gradual passage* to a reduced form involves less labour and calculation than formerly for the question of compounding substitutions does not arise. We take two examples to illustrate the point.

Ex. For the forms (63, 25, 10), (61, —103, 174) of determinant D=—5, the equation (22) becomes

$$x^2+5y^2=3843$$

which possesses no integral solutions for it reduces to the impossible conguence

$$x^2 \equiv 3 \pmod 5$$

because 3 is a quadratic non-residue of 5. Hence the forms are not equivalent.

Worked out by the ordinary method, the process is

(63, 25, 10) ⤷(10, 5, 3) ⤷(3, 1, 2) ⤷(2, 1, 3)

(61,—103, 174) ⤷(174,—71, 29) ⤷(29, 13, 6) ⤷(6,—1, 1) ⤷(1, 0, 5).

The reduced froms are non-equivalent and therefore the proposed forms are so. It may be observed that the computation of the 7 adjacent forms in this case entails an amount of labour which is not in keeping with the almost instantaneous solution arrived at by our method. Similarly, because integral solutions do not obviously exist for the equation $x^2+11y^2=90$, the forms (3, 2, 5) and (30, -53, 94) are non-equivalent.

Thus when the determinant D is negative, it appears to me that for the reasons set forth above and the illustrations adduced in support of them, the present method is much more advantageous than, and I trust also superior to, the existing method of *gradual passage* to reduced forms.

4. But when the determinant D is positive, we are faced with the solution of the equation

$$x^2-Dy^2=aa' \tag{4}$$

which admits of no solution or an infinite number of solutions. Even when they exist there seems to be no direct and independent method of determining them unless aa' happens to be the denominator of the complete quotients obtained in the process of converting $\sqrt{D}$ into a recurring continued fraction. Moreover, there may be more sets of solutions than one, each set including an infinite number which are all deducible from a common formula but those of different sets being not so deducible. And we cannot also rule out the probability that solutions belonging to one set make all the numbers (5) integers, whereas those of another set do not. Lastly, when the equation (4) is impossible, we cannot dismiss it as unsoluble as readily and easily as we did when the determinant D was taken to be negative. Thus although from a theoretical point of view the present method is complete, the practical difficulty is enormously increased when we pass from the negative to the positive determinant. But it cannot be ignored that the difficulty with the existing method also increases at the same time though to a lesser extent.

The fundamental difficulty lies in the solution of an equation such as

$$\ldots \quad x^2-Dy^2=m, \tag{23}$$

where D is a positive integer and m any integer positive or negative, and when it is once overcome, the other point, *viz.*, to decide the integrality of the numbers (5) is comparatively simple. In what follows we shall obtain several results in connection with the latter

point which will in some respects at any rate lessen the difficulty referred to in the preceding paragraph.

Let (T', U') denote the fundamental solution of the equation (4), *i.e.*, a solution for which T' and U' have the least possible positive integral values; then it is at once seen that

(24) ... $(T'T_n + DU'U_n, \quad T'U_n + T_nU')$, $n = \pm 1, \pm 2, \pm 3......$

is also a solution of the equation (4),

where

$$(T + U\sqrt{D})^n = T_n + U_n\sqrt{D},$$

(T, U) as before denoting the fundamental solution of the equation

(2) $$x^2 - Dy^2 = 1.$$

We shall now prove the following:—

If the solution (T', U') of the equation (4) makes

(25)................$T' - bU' \equiv 0 \pmod{a}$, $T' + b'U' \equiv 0 \pmod{a'}$, $(b' - b)T' + (D - bb')U' \equiv 0 \pmod{aa'}$,

then the same will be true when we replace the solution (T', U') by any other included in (24).

We have

$$T'T_n + DU'U_n - b(T'U_n + T_nU')$$

$$= T_n(T' - bU') - U_n(bT' - DU')$$

$$\equiv -U_n\{bT' - (b^2 - ac)U'\} \pmod{a} \text{ by } (25)$$

$$\equiv -bU_n(T' - bU') \pmod{a}$$

$$\equiv 0 \pmod{a}.$$

Similarly, employing

$$T' + b'U' \equiv 0 \pmod{a'}$$

we prove that

$$T'T_n + DU'U_n + b'(T'U_n + T_nU') \equiv 0 \pmod{a'}.$$

Lastly,

$$(b'-b)(\mathrm{T'T}_n+\mathrm{DU'U}_n)+(\mathrm{D}-bb')(\mathrm{T'U}_n+\mathrm{T}_n\mathrm{U}')$$

$$=\mathrm{T}_n\{(b'-b)\mathrm{T}'+(\mathrm{D}-bb')\mathrm{U}'\}+\mathrm{U}_n\{(\mathrm{D}-bb')\mathrm{T}'+(b'-b)\mathrm{DU}'\}$$

$$\equiv\{(\mathrm{D}-bb')\mathrm{T}'+(b'-b)\mathrm{DU}'\}\mathrm{U}_n \pmod{aa'} \text{ by (25)}$$

$$\equiv[\mathrm{D}(\mathrm{T}'-b\mathrm{U}')-b'\{b\mathrm{T}'-(b^2-ac)\mathrm{U}'\}]\mathrm{U}_n \pmod{aa'}$$

$$\equiv\{\mathrm{D}(\mathrm{T}'-b\mathrm{U}')-bb'(\mathrm{T}'-b\mathrm{U}')-b'ac\mathrm{U}'\}\mathrm{U}_n, \pmod{aa'}$$

$$\equiv\{(\mathrm{D}-bb')\mathrm{T}'-b(\mathrm{D}-bb')\mathrm{U}'-ac(\mathrm{T}'+b'\mathrm{U}')+ac\mathrm{T}'\}\mathrm{U}_n \pmod{aa'}$$

$$\equiv\{(\mathrm{D}-bb')\mathrm{T}'+b(b'-b)\mathrm{T}'+ac\mathrm{T}'\}\mathrm{U}_n \pmod{aa'}$$

$$\because\ a(\mathrm{T}'+b'\mathrm{U}')\equiv 0,\quad (b'-b)\mathrm{T}'\equiv-(\mathrm{D}-bb')\mathrm{U}' \pmod{aa'} \text{ by (25)}$$

$\equiv 0 \pmod{aa'}$, $\because\ \mathrm{D}=b^2-ac$.

Thus we get what is required.

It must not, however, be supposed that every solution of the equation (4) makes all the three numbers (5) integral. For we can easily verify that (22, 4), (54, −12) are both solutions of the equation $x^2-19y^2=180$ and whereas the first of them makes the numbers (5) respectively equal to 3, −1, −1 we get the same to be severally equal to 11, $-\frac{19}{3}$, 7 from the second, the forms taken being respectively (6, 1, −3) and (30, −13, 5).

Now it is easily seen that (t, u) being any solution of the equation (4), we have

$$\frac{t-bu}{a}\cdot\frac{t+b'u}{a'}-1=\frac{(b'-b)t+(\mathrm{D}-bb')u}{aa'}\cdot u,$$

so that if the solution (t, u) is such as to make each of

$$\frac{t-bu}{a},\ \frac{t+b'u}{a'}$$

an integer, then

$$\frac{(b'-b)t+(\mathrm{D}-bb')u}{aa'}\cdot u$$

becomes an integer at the same time. If, therefore, u is prime to aa', it follows that

$$(26) \qquad \frac{(b'-b)t+(D-bb')u}{aa'}$$

is also an integer and thus we have the following result:—

If (t, u) is a primitive solution (i.e., a solution for which t and u are prime to each other) of the equation (4) makes both $\frac{t-bu}{a}$ and $\frac{t+b'u}{a'}$ integral, then will also (26) be an integer.

Cor. If aa' does not contain a square factor, then a solution of the equation (4) is certainly primitive and hence we conclude

If aa' contains no square factor, then every solution of the equation (4) which makes the first and the third of the numbers (5) integral makes also the second integral.

Let us consider two forms (a, b, c) and (a', b', c') of the same positive determinant D. The product aa' will not contain a square factor if neither of a and a' contains one and a is relatively prime to a'. We have

$$\begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}(a, b, c)=(a\lambda^2+2b\lambda+c, -b-\lambda a, a),$$

$$\begin{pmatrix} \lambda' & -1 \\ 1 & 0 \end{pmatrix}(a', b', c')=(a'\lambda'^2+2b'\lambda'+c', -b'-\lambda'a', a')$$

Now a few trials with values of λ and λ' (in most cases $\lambda=0, \pm 1$, $\lambda'=0, \pm 1$ will do) will ensure that none of

$$a\lambda^2+2b\lambda+c, \ a'\lambda'^2+2b'\lambda'+c'$$

contain a square factor and both are relatively prime to each other so that in case of necessity we can replace one or both the forms (a, b, c), (a', b', c') by their equivalent forms according to this method. In this way the product of the first co-efficients can be freed from a square factor.

We shall say nothing regarding the solution of the equation (23) except that when m is not a denominator of the complete quotients obtained in the process of converting $\sqrt{D}$ into a recurring continued

fraction, it seems best to obtain a solution by inspection and trial. We then ascertain whether or no it makes the first and the third of the numbers (5) integers and in case it is so, determine the substitution S converting (a, b, c) into (a', b', c'). If m is a quadratic non-residue of D, then clearly the equation is impossible.

We conclude this paper by giving an example which will demonstrate the comparative advantages and disadvantages of the two methods when the determinant D is positive.

Ex. Examine by the two methods the equivalence of the forms (52, 69, 91), (265, 192, 139), of determinant D=29.

A solution of the equation $x^2-29y^2=52\cdot265$ is found out to be (2367, 439) and it makes the integers (5) respectively equal to -537, 400, 327 so that we get

$$\begin{pmatrix} -537 & 400 \\ 439 & 327 \end{pmatrix}(52,\ 69,\ 91,)=(265,\ 192,\ 139).$$

The following indicates the process to be adopted by the ordinary method.

$(52,\ 69,\ 91) \backsim (91,-69,\ 52) \backsim (52,-35,\ 23) \backsim (23,-11,\ 4) \backsim (4,\ 3,-5),$

$(265,\ 192,\ 139) \backsim (139,-53,\ 20) \backsim (20,-7,\ 1) \backsim (1,\ 5,-4).$

$(4,\ 3,-5) \backsim (-5,\ 2,\ 5) \backsim (5,\ 3,-4) \backsim (-4,\ 5,\ 1) \backsim (1,\ 5,-4).$

The form (52, 69, 91) is converted into $(4, 3, -5)$ by the substitution

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}=\begin{pmatrix} -3 & 4 \\ 2 & -3 \end{pmatrix}.$$

Secondly, the substitution which converts (265, 192, 139) into $(1, 5, -4)$

is. $$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}=\begin{pmatrix} -3 & 5 \\ 4 & -7 \end{pmatrix}$$

Thirdly, the substitution which converts $(4, 3, -5)$ into $(1, 5, -4)$ is

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -10 \end{pmatrix}=\begin{pmatrix} 3 & 31 \\ 5 & 52 \end{pmatrix}$$

We thus get

$$\begin{pmatrix} -3 & 4 \\ 2 & -3 \end{pmatrix}(52,\ 69,\ 91)=(4,\ 3,-5)\ ;\begin{pmatrix} 3 & 31 \\ 5 & 52 \end{pmatrix}(4,\ 3,-5)=(1,\ 5,-4) \text{ and}$$

$$\begin{pmatrix} -7 & -5 \\ -4 & -3 \end{pmatrix}(1, 5, -4)=(265, 192, 139)$$

$$\therefore \begin{pmatrix} -3 & 4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 3 & 31 \\ 5 & 52 \end{pmatrix}\begin{pmatrix} -7 & -5 \\ -4 & -3 \end{pmatrix}(52, 69, 91)=(265, 192, 13$$

$$\text{or } \begin{pmatrix} -537 & -400 \\ 439 & 327 \end{pmatrix}(52, 69, 91)=(265, 192, 139)$$

Thus we get the same result.

The solution of the equation $x^2-Dy^2=m$ can undoubtedly reduced to one in which the right side m is less than $\sqrt{D}$.*

But it requires a series of transformations. Also it can be made depend upon the solutions of equations such as $x^2-Dy=m_i$ whe $m_i,(i=1, 2, \ldots\ldots)$ denotes factors of m resolved so as to make th product equal to m. In any case the task is laborious. But it is r less so by the existing method as will be sufficiently clear from t work exhibited above. The chief defect of the present method appea to be its dependence upon an equation whose solution when D positive, is often troublesome and for which there is no general formu known.† The chief excellence of the classical method is its elementa character and the absence of any uncertain step in the procedure b it fails to give a general expression such as we have obtained here.

* Chrystal's *Algebra*, Pt. II, p. 482f.

† Lagrange has proved that if the equation $x^2-Dy^2=m$, $m<\sqrt{D}$ is soluble, can be reduced to one in which $\frac{x}{y}$ is a convergent to the continued fraction for $\sqrt{}$ (H. J. S. Smith, *Collected Math. Papers*, Vol. I, Oxford, 1894, p. 199).

On Stresses in Circular Rings under the Action of Isolated Forces on the Rim.

BY

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1. In the paper on "The Stress in a Circular Ring."* Prof. Filon has given a general method of deducing the stress resultants at a cross section of a plane circular ring in terms of loads applied at the rims. Particular cases which he has discussed mainly relate to the action of equilibrating forces on the inner rim. The object of this paper is to find directly and by a different method the stresses in a plane circular ring when the outer rim as well as the inner rim is acted on by isolated forces. The stress function in the following cases has been determined.

(1) The ring acted on by equal and opposite forces on the outer boundary operating along a chord.

(2) The ring diametrically compressed.

(3) The ring acted on by three forces in equilibrium.

(4) The ring acted on by equal and opposite forces on the inner boundary.

(5) The ring acted on by equal and opposite forces on the two boundaries one operating on each.

It is believed that the method and most of the results are new.

2. Following Filon, let us assume that the ring consists of a thin plate bounded by concentric circles and that it is in a state of generalised plane stress. In that case we know that the average stresses $\widehat{rr}$, $\widehat{r\theta}$ and $\widehat{\theta\theta}$ at any point (r, θ) are given by the relations

$$\widehat{rr}=\frac{1}{r}\frac{\partial \chi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \chi}{\partial \theta^2},$$

* Selected Engineering papers published by the Institution of Civil Engineers,

$$\widehat{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \chi}{\partial \theta}\right), \qquad \dots \quad (1)$$

$$\widehat{\theta\theta} = \frac{\partial^2 \chi}{\partial r^2},$$

where χ is the stress function which satisfies the relation

$$\nabla_1^4 \chi = 0 \qquad \dots \quad (2)$$

where ∇_1^2 stands for $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

3. **Case I.** *Outer boundary acted on by two equal and opposite forces at the ends of a chord and along it.*

Let two forces F act inwards along O_1O_2 at the points O_1, O_2 on the outer boundary. If O_1O_2 subtends $180° - 2a$ at the centre O of the ring, then the stress function χ_1 corresponding to these forces and yielding no stresses on the external boundary $r=a$ can be written as

$$\chi_1 = \frac{F}{\pi}\left[\frac{r^2 \cos \alpha}{2a} - (r_1\theta_1 \sin \theta_1 + r_2\theta_2 \sin \theta_2)\right] \qquad \dots \quad (3)^*$$

where r_1, r_2, r are the distances of any point P from O_1, O_2 and O respectively and $\theta_1 = \angle PO_1O_2$ and $O_2 = \angle PO_2O_1$.

With reference to the centre O of the ring as origin and the diameter parallel to O_1O_2 as the initial line, the above expression becomes

$$\chi_1 = \frac{F}{\pi}\left[\frac{r^2}{2a}\cos \alpha - (r \sin \theta - a \sin \alpha)\right.$$

$$\left.\times \left\{\tan^{-1}\frac{r \sin (\theta+\alpha)}{a + r\cos (\theta+\alpha)} + \tan^{-1}\frac{r \sin (\theta-\alpha)}{a - r\cos(\theta-\alpha)} - 2\alpha\right\}\right].$$

If the inner boundary be $r=b$, then in the region

$$a > r > b$$

$$\tan^{-1}\frac{r \sin (\theta+\alpha)}{a + r \cos (\theta+\alpha)} = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{r^n}{na^n} \sin n(\theta+\alpha)$$

$$\text{and } \tan^{-1}\frac{r \sin (\theta-\alpha)}{a - r \cos (\theta-\alpha)} = \sum_{n=1}^{\infty} \frac{r^n}{na^n} \sin n(\theta-\alpha).$$

* Love's Elasticity, Fourth Edition, p. 318.

Hence χ_1 can be written as

$$\chi_1 = \frac{Fa}{\pi}\Bigg[-\frac{r^2}{2a^2}\cos\alpha - 2\alpha\sin\alpha + \sin\theta\Bigg\{\frac{2ar}{a}$$

$$+\frac{2r}{a}\sin\alpha\cos\alpha - \frac{r^3}{2a^3}\sin 2\alpha\Bigg\}$$

$$+\sum_{m=1}^{\infty}\Bigg\{\Bigg(\frac{r^{2m}}{(2m-1)\,a^{2m}}\cos(2m-1)\alpha - \frac{1}{2m+1}\,\frac{r^{2m+2}}{a^{2m+2}}\cos(2m+1)\alpha$$

$$-\frac{r^{2n}}{ma^{2m}}\sin\alpha\sin 2m\alpha\Bigg)\cos 2m\theta + \Bigg(\frac{r^{2m+1}}{2ma^{2m+1}}\sin 2m\alpha$$

$$-\frac{1}{2m+2}\,\frac{r^{2m+3}}{a^{2m+3}}\sin(2m+2)\alpha + \frac{2\,r^{2m+1}}{(2m+1)a^{2m+1}}\sin\alpha\cos(2m+1)\alpha\Bigg)$$

$$\times\sin(2m+1)\theta\Bigg\}\Bigg] \qquad \ldots \quad (4)$$

calculating the stresses $\widehat{rr}$, $\widehat{r\theta}_1$ from this function χ_1 with the help of the formulae (1), we get

$$\widehat{rr}_1 = \frac{F}{\pi a}\Bigg[-\cos\alpha - \frac{r}{a}\sin\theta\sin 2\alpha$$

$$+\sum_{m=1}^{\infty}\Bigg\{\Bigg(-2m\frac{r^{2m-2}}{a^{2m-2}}\cos(2m-1)\alpha + 2(m-1)\frac{r^{2m}}{a^{2m}}\cos(2m+1)\alpha$$

$$+2(2m-1)\frac{r^{2m-2}}{a^{2m-2}}\sin\alpha\sin 2m\alpha\Bigg)\cos 2m\theta + \Bigg(-(2m+1)\frac{r^{2m-1}}{a^{2m-1}}\sin 2m\alpha$$

$$+(2m-1)\frac{r^{2m+1}}{a^{2m+1}}\sin(2m+2)\alpha - 4m\frac{r^{2m-1}}{a^{2m-1}}\sin\alpha\cos(2m+1)\alpha\Bigg)$$

$$\times\sin(2m+1)\theta\Bigg\}\Bigg] \qquad (5)$$

$$\widehat{r\theta}_1 = \frac{F}{\pi a}\Bigg[\frac{r}{a}\cos\theta\sin 2\alpha +$$

$$+\sum_{m=1}^{\infty}\Bigg\{\Bigg(2m\frac{r^{2m-2}}{a^{2m-2}}\cos(2m-1)\alpha - 2m\frac{r^{2m}}{a^{2m}}\cos(2m+1)\alpha - 2(2m-1)$$

$$\times \frac{r^{2m-2}}{a^{2-2}} \sin\alpha \sin 2m\alpha\Big) \sin 2m\theta + \Big(-(2m+1)\frac{r^{2m-1}}{a^{2m-1}} \sin 2m\alpha$$

$$+(2m+1)\frac{r^{2m+1}}{a^{2m+1}} \sin(2m+2)\alpha - 4m\frac{r^{2m-1}}{a^{2m-1}} \sin\alpha \cos(2m+1)\alpha\Big)$$

$$\times \cos(2m+1)\theta\Big\}\Big] \quad \ldots \quad (6)$$

The values of these stresses on the boundary $r=b$, are

$$(\widehat{rr}_1)_{r=b} = b_0 + b_1 \sin\theta + \Sigma_{m=1}^{\infty}\{b_{2m}\cos 2m\theta + b'_{2m+1}\sin(2m+1)\theta\},$$

$$(\widehat{r\theta}_1)_{r=b} = c_1 \cos\theta + \Sigma_{m=1}^{\infty}\{c_{2m}\sin 2m\theta + c'_{2m+1}\cos(2m+1)\theta\}$$

where $b_0 = -\frac{F}{\pi a}\cos\alpha$; $b_1 = -\frac{F}{\pi a} q \sin 2\alpha$, ($q$ being equal to $\frac{b}{a}$)

$$b_{2m} = 2q^{2m-2}\,[-m\cos(2m-1)\alpha + (m-1)q^2\cos(2m+1)\alpha + (2m-1)\sin\alpha\sin 2m\alpha]$$

$$b'_{2m+1} = q^{2m-1}\,[-(2m+1)\sin 2m\alpha + (2m-1)q^2\sin(2m+2)\alpha - 4m\sin\alpha\cos(2m+1)\alpha]$$

$$c_1 = \frac{F}{\pi a} q \sin 2\alpha$$

$$c_{2m} = 2q^{2m-2}\,[m\cos(2m-1)\alpha - mq^2\cos(2m+1)\alpha - (2m-1)\sin\alpha\sin 2m\alpha]$$

$$c'_{2m+1} = q^{2m-1}\,[-(2m+1)\sin 2m\alpha + (2m+1)q^2\sin(2m+2)\alpha - 4m\sin\alpha\cos(2m+1)\alpha] \quad \ldots \quad (7)$$

Let $\chi = \chi_1 + \chi_2$

where χ_2 satisfies the equation $\nabla_1{}^4\chi_2 = 0$,

and is such that the stresses $\widehat{rr}$ and $\widehat{r\theta}$ calculated from it vanish at the boundary $r=a$ and balance those stresses produced by χ_1 on the boundary $r=b$.

Excluding the terms that lead to multiple valued displacements in the ring a simplified form of the function can be written as

$$\chi_2=\frac{Fa}{\pi}\Big[A_0\Big(\log r-\frac{r^2}{2a^2}\Big)+B\Big(\frac{r^3}{a^3}-\frac{a}{r}\Big)\sin\theta$$

$$+\sum_{m=1}^{\infty}\Big(A_{2m}\Big\{-(2m+1)\frac{r^{2m}}{a^{2m}}+2m\frac{r^{2m+2}}{a^{2m+2}}+\frac{a^{2m}}{r^{2m}}\Big\}$$

$$+B_{2m}\Big\{-\frac{2m}{a^{2m}}r^{2m}+(2m-1)\frac{r^{2m+2}}{a^{2m+2}}+\frac{a^{2m-2}}{r^{2m-2}}\Big\}\Big)\cos 2m\theta$$

$$+\sum_{m=1}^{\infty}\Big(C_{2m+1}\Big\{-(2m+2)\frac{r^{2m+1}}{a^{2m+1}}+(2m+1)\frac{r^{2m+3}}{a^{2m+3}}+\frac{a^{2m+1}}{r^{2m+1}}\Big\}$$

$$+D_{2m+1}\Big\{-\frac{(2m+1)r^{2m+1}}{a^{2m+1}}+2m\frac{r^{2m+3}}{a^{2m+3}}+\frac{a^{2m-1}}{r^{2m-1}}\Big\}\Big)\sin(2m+1)\theta\Big] \quad \dots \quad (8)$$

Calculating $\widehat{(rr)}_2$ and $\widehat{(r\theta)}_2$ out of this function it is easily found that

$$\widehat{(rr)}_2=\widehat{(r\theta)}_2=0 \text{ on the boundary } r=a.$$

The stress due to χ_1 and χ_2 will vanish at the internal boundary $r=b$ if

$$A_0=\frac{q^2}{1-q^2}\cos\alpha$$

$$B=\frac{q^4}{-2(1-q^4)}\sin 2\alpha$$

$$A_{2m}=\frac{[4m^2q^{2m-2}-(2m+1)(2m-2)q^{2m}-(2m+2)\,q^{-2m}]}{(2m+1)\quad Q_{2m}}c_{2m}$$

$$-\frac{[-4m^2q^{2m-2}+2m(2m+1)q^{2m}-2mq^{-2m}]}{(2m+1)\quad Q_{2m}}b_{2m}$$

$$B_{2m}=\frac{[-2m(2m-1)\,q^{2m-2}+4m^2q^{2m}-2mq^{-(2m+2)}]}{(2m-1)\quad Q_{2m}}\,b_{2m}$$

$$-\frac{[2m\,(2m-1)q^{2m-2}-2m\;(2m-2)\;q^{2m}-2mq^{-(2m+2)}\,]}{(2m-1)\quad Q_{2m}}\,c_{2m}$$

$$Q_{2m}=-4mq^{-(4m+2)}\,[1-4m^2q^{4m-2}(1-2q^2+q^4)-2q^{4m}+q^{8m}].$$

C_{2m+1} and D_{2m+1} are obtained by putting $(2m+1)$ for $2m$ in the expressions for A_{2m} and B_{2m}, χ being determined, the stresses can be easily calculated.

4. **Case II.** *A ring, diametrically compressed.*

If a ring be compressed by two equal and opposite forces F acting along a diameter, the stress can be deduced from the above expressions by putting $\alpha=0$.

The simplified value of χ_1 is given by

$$\chi_1=\frac{F}{\pi}\left[\frac{r^2}{2a}-r\;\sin\theta\tan^{-1}\frac{2ar\sin\theta}{a^2-r^2}\right]$$

$$=\frac{Fa}{\pi}\left[-\frac{r^2}{2a^2}+\Sigma_{m=1}^{\infty}\left\{\frac{r^{2m}}{(2m-1)a^{2m}}-\frac{r^{2m+2}}{(2m+1)a^{2m+2}}\right\}\cos 2m\theta.\right] \quad (9)$$

It is apparent this value of χ_1 gives *zero* values of f the normal and shear stresses on the boundary $r=a$ except at the points where the forces act.

For χ_2 let us assume the expression

$$\frac{Fa}{\pi}\left[A_0\left(\log r-\frac{r^2}{2a^2}\right)+\Sigma_{m=1}^{\infty}\left(A_{2m}\left\{-(2m+1)\frac{r^{2m}}{a^{2m}}+2m\frac{r^{2m+2}}{a^{2m+2}}\right.\right.\right.$$

$$\left.\left.\left.+\frac{a^{2m}}{r^{2m}}\right\}+B^{2m}\left\{-2m\frac{r^{2m}}{a^{2m}}+(2m-1)\frac{r^{2m+2}}{a^{2m+2}}+\frac{a^{2m-2}}{r^{2m-2}}\right\}\right)\cos\;2m\theta\right].$$

Then the boundary $r=b$ will also be free from stress if

$$A_0=\frac{q^2}{1-q^2},\left(q\text{ being equal to }\frac{b}{a}\text{ as, before}\right).$$

$$A_{2m}=q^{4m}\frac{(2m+1)-2mq^2-q^{4m}}{(2m+1)Q'_{2m}}$$

$$B_{2m}=-q^{4m-2}\times\frac{2m-(2m-1)\ q^2-q^{4m+2}}{(2m-1)Q'_{2m}}$$

where $Q'_{2m}=1-4m^2q^{4m-2}(1-2q^2+q^4)-2q^{4m}+q^{8m}$.

5. **Case III.** *Three forces acting on the outer boundary.*

If three forces acting at the points O_1, O_2 and O_3 on the outer boundary be such as can keep a rigid body in equilibrium, then we can resolve these forces into pairs of equal forces acting along the chords O_1O_2, O_2O_3 and O_3O_1. Hence with the help of the results obtained in case I, we can write down the values of the stress.

6. **Case IV.** *The inner boundary acted on by two equal and opposite forces applied at diametrically opposite points.*

This case has been considered by Prof. Filon. Since the problem is an important one it is worth while to try it by the method adopted in the previous cases.

If a single force F acts at the point $r=b$, $\theta=0$ in the positive direction of the initial line then the corresponding stress function is $-\frac{F}{\pi}r_1\theta_1\sin\theta_1$, (r_1, θ_1) being the co-ordinates of any point with reference to the point of application as the pole. Similarly, if another force acts at the point $r=b$, $\theta=\pi$ in the opposite direction, the corresponding stress function will be $-\frac{F}{\pi}r_2\theta_2\sin\theta_1$.

By simple geometry, it can be seen that if r, θ be the co-ordinates of any point with reference to the centre as the pole, then

$$r_1\sin\theta_1=r_2\sin\theta_2=r\sin\theta,$$

$$\theta_1=\tan^{-1}\frac{b\sin\theta}{r+b\cos\theta}+\theta,$$

$$\text{and } \theta_2=\tan^{-1}\frac{b\sin\theta}{r-b\cos\theta}+\pi-\theta.$$

Then the stress function χ_1 which will yield zero stresses over the boundary $r=b$ except at the points of application of the forces

and representing the effect of these forces elsewhere can the written as

$$\chi_1=\frac{F}{\pi}\left[\frac{r^2}{2b}-\pi r\sin\theta-r\sin\theta\tan^{-1}\frac{2br\sin\theta}{r^2-b^2}\right]$$

$$=\frac{Fb}{\pi}\left[\frac{r^2}{2b^2}-\frac{\pi r}{b}\sin\theta-\frac{1}{2}+\Sigma_{m=1}^{\infty}\left\{\frac{b^{2m-2}}{(2m-1)r^{2m-2}}-\frac{b^{2m}}{(2m+1)r^{2m}}\right\}\cos 2m\theta\right] \quad \ldots \quad (10)$$

As in Case 2, let us assume

$$\chi_2=\frac{Fb}{\pi}\left[A_0\left(\log r-\frac{r^2}{2b^2}\right)\right.$$

$$+\Sigma_{m=1}^{\infty}\left\{A_{2m}\left[-(2m+1)\frac{r^{2m}}{b^{2m}}+2m\frac{r^{2m+2}}{b^{2m+2}}+\frac{b^{2m}}{r^{2m}}\right]\right.$$

$$\left.\left.+B_{2m}\left[-2m\frac{r^{2m}}{b^{2m}}+(2m-1)\frac{r^{2m+2}}{b^{2m+2}}+\frac{b^{2m-2}}{r^{2m-2}}\right]\right\}\cos 2m\theta \right] \quad (11)$$

Putting $\chi=\chi_1+\chi_2$ and calculating the stresses by the formulæ, we find that they will vanish on the outer boundary $r=a$ if

$$A_{2m}=-\frac{q^{4m-2}[4m^2(1-q^2)^2+2m(1-q^2)+q^2(1-q^{4m})]}{(2n+1)Q_{2n}},$$

$$B_{2m}=q^{4m-2}\frac{[4m^2(1-q^2)^2+2mq^2(1-q^2)+q^2(1-q^{4m})]}{(2m-1)Q_{2n}},$$

$$Q'_{2n}=(1-q^{4m})^2-4m^2q^{4m-2}(1-q^2)^2,$$

$$\text{and } A_0=-\frac{1}{1-q^2}. \quad (12)$$

7. **Case V.** *The inner and outer rims acted on by equal and opposite forces.*

If at the point $r=b$, $\theta=0$ on the inner rim a force F acts in the positive direction, the corresponding stress function is given by

$-\frac{F}{\pi} r_1\theta_1 \sin\theta_1$ with reference to the point of application of the force as the pole.

With reference to the centre of the ring, it becomes

$$\chi_1 = -\frac{F}{\pi}\left[r\sin\theta\left(\theta+\tan^{-1}\frac{b\sin\theta}{r-b\cos\theta}\right)\right].$$

The term $r\theta\sin\theta$ gives multiple valued displacements in the region bounded by the two circles and its effect can be counteracted by subtracting a term $r\nu\cos\theta\log r$ which does not alter the magnitude of the force ν being equal to $\frac{1}{2}(1-\sigma)$ where σ is Poisson's ratio.

Hence $\chi_1 = -\frac{F}{\pi}\left[r\theta\sin\theta - \nu r\cos\theta\log r + r\sin\theta\tan^{-1}\frac{b\sin\theta}{r-b\cos\theta}\right].$

Expanding the function $\tan^{-1}\frac{b\sin\theta}{r-b\cos\theta}$ in powers of $\frac{b}{r}$ in the region $r>b$, we get

$$\chi_1 = -\frac{Fb}{2\pi}\left[\frac{2r\theta}{b}\sin\theta - 2\nu\frac{r}{b}\cos\theta\log r + \frac{b\cos\theta}{2r}\right.$$

$$\left. + \sum_{n=2}^{\infty}\left(\frac{b^n}{(n+1)r^n} - \frac{b^{n-2}}{(n-1)r^{n-2}}\right)\cos n\theta\right] \quad \ldots \quad (13)$$

If another force F act at the point $(a, 0)$ opposite to the above force then the corresponding stress function χ_2 will be $-\frac{F}{\pi} r_2\theta_2\sin\theta_2$ which, if referred to the centre of the ring, becomes

$$\chi_2 = -\frac{Fa}{2\pi}\left[2+\frac{r}{a}\cos\theta + \sum_{n=2}^{\infty}\left\{\frac{1}{n+1}\frac{r^{n+2}}{a^{n+2}} - \frac{1}{(n-1)}\frac{r^n}{a^n}\right\}\cos n\theta.\right] \quad (14)$$

Calculating the stresses from (13) and (14) we find that the stresses on the boundaries can be written as follows :

R_a = the radial stress on $(r=a)$

$$=-\frac{F}{2\pi}\left[a_0+\sum_{n=1}^{\infty} a_n \cos n\theta\right],$$

R_b = the radial stress on $(r=b)$

$$=-\frac{F}{2\pi}\left[b_0+\sum_{n=1}^{\infty} b_n \cos n\theta\right].$$

The shear stresses on those boundaries are

$$-\frac{F}{2\pi}\left[\sum_{n=1}^{\infty} c'_n \sin n\theta\right] \text{ and } -\frac{F}{2\pi}\left[\sum_{n=1}^{\infty} d'_n \sin n\theta\right] \text{ respectively}$$

where $a_0=b_0=\dfrac{2}{a}$.

$$a_1=\frac{1}{a}-\frac{1}{b}\left[2(2-\nu)q+q^3\right]$$

$$b_1=\frac{q}{a}-\frac{1}{b}\left[2(2-\nu)+1\right].$$

$$c_1'=\frac{1}{a}-\frac{1}{b}\left(2\nu q+q^3\right).$$

$$d'_1=\frac{q}{a}-\frac{1}{b}\left(2\nu+1\right).$$

and for $n>1$,

$$a_n=\frac{2}{a}+\frac{1}{b}\left[(n+2)q^n-nq^{n+2}\right],$$

$$b_n=\frac{2}{b}+\frac{1}{a}\left[nq^{n-2}-(n-2)q^n\right],$$

$$c_n'=\frac{n}{b}\left[q^{n+2}-q^n\right],$$

$$d'_n=\frac{n}{a}\left[q^n-q^{n-2}\right]$$

In this case following the general method of Filon* we can write down

$$\chi=\chi_1+\chi_2+\chi_3$$

where

$$\chi_3=\frac{F}{2\pi}\Big[A_0\log r+C_0r^2+\left(B_1r^3+\frac{C_1}{r}\right)\cos\theta$$

$$+2D_1(\nu r\cos\theta\log r-r\theta\sin\theta)$$

$$+\sum_{n=2}^{\infty}(A_nr^n+B_nr^{n+2}+C_nr^{-n}+D_nr^{-n+2})\cos n\theta\Big].$$

The constants of the above expression can be determined from the condition that the resultant stresses on the boundaries may be zero.

If $\eta=\dfrac{1}{2(1-\nu)}$ then it can be seen from the results worked out by Filon that

$$A_0=0,\qquad C_0=\frac{1}{2}a_0,$$

$$B_1a=\{(1-2\eta)(a_1-b_1q^3)+(3-2\eta)(c'_1-d'_1q^3\}/8(1-\eta)(1-q^4)$$

$$C_1/a^3=\{(1-2\eta)(a_1q^4-b_1q^3)+(3-2\eta)(c'_1q^4-d'_1q^3)\}/8(1-\sigma)(1-q^4)$$

$$D_1=-\frac{a(a_1-c'_1)}{4(1-\eta)}=\frac{b(b_1-d'_1)}{4(1-\eta)}.$$

and for $n>1$

$$A_na^{n-2}=\xi_ng_n-\eta_nf_n,$$

$$B_na^n=\xi_nf_n-\zeta_ng_n,$$

$$C_na^{-n}b^{-2}=\xi_nh_n-\zeta_nk_n,$$

$$D_na^{2-n}=\xi_nk_n-\eta_nh_n,$$

where $f_n=n\{a_n-c'_n-q^n(b^n-d_n')\}$,

$$g_n=na_n-(n+2)c_n'-q^{n+2}\{nb_n-(n+2)d_n'\},$$

$$h_n = q^n\{nb^n + (n-2)d_n'\} - q^{2n-2}\{na_n + (n-2)\}c_n',$$

$$k_n = nq^n(b_n + d_n') - nq^{2n}(a_n + c_n'),$$

$$\xi_n = \tfrac{1}{2}(1-q^{2n})/nQ_n, \quad \eta_n = \tfrac{1}{2}(1-q^{2n+2})/(n-1)Q_n,$$

$$\zeta_n = (1-\tfrac{1}{2}q^{2n-2})/(n+1)Q_n,$$

where $Q_n = (1-q^{2n})^2 - n^2q^{2n-2}(1-q^2)^2$.

In a similar method some other cases can be solved. In some cases, *e.g.*, in the case of the diametrically compressed ring, it is apparent from the values of the constants that suffieciently approximate values can be obtained by retaining the first few terms if q or $\frac{b}{a}$ be small.

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4

ON THE SUPPOSED INDEBTEDNESS OF BRAHMAGUPTA TO *Chiu-chang Suan-shu*

BY

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Some of the modern writers on the history of mathematics are under the impression that the eminent Hindu astronomer and mathematician Brahmagupta (628 A.D.) has taken a certain problem from the great Chinese mathematical classic *Chiu-chang Suan-shu*, which was composed originally by Chang T'sang (died 152 B.C.) and subsequently revised and enlarged by Ching Ch'ou-ch'ang (*c*. 50 B. C.). That is indeed a very wrong impression and the present note aims at setting it right.

We do not know who is originally responsible for making that erroneous statement. Indeed it will not serve any useful purpose for the cause of Hindu mathematics to trace the statement to its origin. It is, however, found to have been repeated, with more or less emphasis, by more than one modern historian of mathematics. And the object of almost all of them was to prove the dependence of Hindu mathematics on Chinese mathematics. Professor David Eugene Smith wrote in 1912: "While Sanskrit was known there [in China] very early, and by about 800 A.D. was even taught in Japan (through the writings of the great scholar, Kōbō Daishi), there is nothing in the mathematics of either country that shows dependence upon any known works of Hindu scholars. On the contrary, it would seem that Brahmagupta, who wrote in Ujjain in the seventh century, was indebted for at least one of his problems to the great Chinese classic, the Chiu-chang Suan-shu."[1] The specific problem referred to in this statement is this:[2]

[1] D. E. Smith, "Chinese Mathematics," *The Popular Science Monthly*, Vol. 80, 1912, pp. 597-601.

[2] Y. Mikami, *Development of Mathematics in China and Japan*, Leipzig, 1913, p. 23; hereafter referred to as Mikami, *Chinese Mathematics*.

"There is a bamboo 10 feet high, the upper end of which being broken down reaches the ground at 3 feet from the stem ; what is the height of the break ?"

Thirteen years later, Smith[1] went a step further and wrote: "The problem about the tree is found in various Hindu mathematical works after the time of Aryabhata." Professor Yoshio Mikami,[2] an well-known authority on the history of Chinese mathematics remarked in 1913 : "This problem highly interests us, because it is stated to be contained also in Brahmagupta's work. It is certain that the Indian learning exceedingly influenced Chinese thought, but at the same time the discoveries made in China must have touched the eyes of Hindu scholars. If it were not a mere coincidence that the problem under consideration appeared in the same form in the treatises of the two neighbouring countries, one of the two must have borrowed it from the other ; and this may be an instance of the Chinese productions transferred to the enlightened land in the south; for Brahmagupta appeared full six or seven centuries subsequent to the compilation of the Chinese treatise which we are just describing."[3] Though Mikami was thus very cautious in making his observations, two years later Kaye [4] exaggerated his misnotion by stating that "this example occurs in every Indian work after the sixth century." Professor Florian Cajori[5] has closely followed Kaye.

The real fact is that neither the example referred to by all those writers, nor any other of its kind, occurs in the works of Brahmagupta. It seems strange that such an erroneous impression could prevail so long without being challenged.

Problems of that kind, not exactly that one, are found in India in the *Gaṇita-sāra-saṁgraha*[6] of Mahāvīra (850 A.D.), the commentary of Pṛthudakasvāmī (860) on the *Brāhma-sphuṭa-siddhānta*[7] of

[1] D. E. Smith, *History of Mathematics*, in two volumes ; Vol. I, p. 139.

[2] *Loc. cit.*, p. 23.

[3] Reference is to the *Chiu-chang Suan-shu.*

[4] G. R. Kaye, *Indian Mathematics*, Calcutta, 1915, p. 39.

[5] F. Cajori, *History of Mathematics*, 2nd edition, New York, 1922, p. 97; also pp. 71 f.

[6] vii. 191½-192½.

[7] H. T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bháscara*, London, 1817, p. 309 fn. This work will be hereafter referred to as Colebrooke, *Hindu Algebra.*

Brahmagupta, *Līlāvatī* [1] and *Bījagaṇita*[2] of Bhāskara II (1150).[3] The solution given by them is practically the same as that given by Ching Ch'ou-ch'ang, namely,

$$\text{Height of break} = \frac{(\text{full height})}{2} - \frac{(\text{distance from stem})^2}{2\,(\text{full height})}$$

Of course it could not be otherwise. But the form in which the Hindu mathematicians expressed the final result, or obtained it, is different. Thus according to Mahāvīra,

$$\text{Height of break} = \tfrac{1}{2}\{(\text{full height})^2 - (\text{distance from stem})^2\} \div (\text{full height}).$$

Pṛthudakasvāmī has obtained the solution by a cumbrous method with the help of certain properties of circle, while other scholars have done with the properties of the right-angled triangle. He supposes a circle passing through the point where the tip reaches the ground, with its centre at the point of break and its plane as the plane of the two broken portions. Then the distance from the stem is a semi-chord of this circle. Therefore

$$\text{Diameter of the circle} = (\text{full height}) + \frac{(\text{distance from stem})^2}{\text{full height}}$$

and

$$\text{Height of break} = (\text{full height}) - (\text{semi-diameter}).$$

In the *Bījagaṇita*, the height of break (x) is obtained by solving the following equation:

$$x^2 + b^2 = (c - x)^2$$

where c is the full height and b the distance from the stem.

Now it may be asked whether any of the Hindu scholars who set a problem of that kind, derived it from the Chinese source. It is not easy at the present undeveloped state of the history of the development of the science of mathematics in these two countries to

[1] *Līlāvatī*, edited by Sudhakara Dvivedi, Benares, 1910, p. 37.

[2] *Bījagaṇita*, edited by Sudhakara Dvivedi and Muralidhara Jha, Benares, 1927, p. 57.

[3] Henceforth we shall designate the celebrated author of *Śiddhānta-śiromaṇi*, *Līlāvatī* and *Bījagaṇita* as Bhāskara II in order to distinguish him from his earlier namesake, the author of *Mahā-Bhāskarīya* and *Laghu-Bhāskarīya*, who will be designated as Bhāskara I. The former, as is well known, was born in 1114 A.D. and the latter flourished in the first half of the sixth century. For further particulars the reader is referred to the writer's forthcoming article, "The two Bhāskaras."

decide the point at issue in a satisfactorily conclusive way. There is, however, one fact which appears to be against the hypothesis of inter-dependence. The two earliest known Hindu writers to record such a problem, Mahāvīra (850) and Pṛthudakasvāmī (860), were contemporary. And they lived in the two furthest ends of this vast sub-continent, the former in the extreme south, in Mysore and the latter very much towards the north, in Kanauja. It seems highly improbable that they had mutual influence, though the similarity of some of their examples is striking.[1] A more natural conclusion will be to presume that either they derived them from a third source or they devised them independently. We do not as yet know of any Hindu writer, anterior to them, who gave such an example. The hypothesis of a foreign source coming to the hands of these contemporary writers living in the two extreme ends of a sub-continent is set with so many other difficulties, hard to explain, that we discard it. We think it more safe to assume that they, Mahāvīra and Pṛthudakasvāmī, devised their problems independently.[2]

The rule of Brahmagupta which seems to have been the source of this error is this:[3]

"The height of the mountain multiplied by an optional multiplier is the full distance of the town. That divided by the optional quantity plus two, is the height of the flight of the other in case of equal journey."

The problem contemplated in the solution given in this rule of Brahmagupta has been pointed out by his commentator, Pṛthudakasvāmī, to be as follows:

"On the top of a certain hill there lived two ascetics. One of them had the super-human power of ascending and travelling in the air. Having ascended up to a certain distance from the summit of

[1] Compare *Gaṇita-sāra-saṁgraha*, Preface, p. xi.

[2] It is interesting to find that another problem of the *Chiu-chang Suan-shu* (Mikami, *loc. cit.*, p. 22) also reappearing in some of the Hindu works: In the midst of a pond there is a lotus, the portion of which above the water-surface is known. Forced by wind or other agencies, it just submerges at a known distance; to find the depth of water and the length of the lotus-stalk. This problem with some alteration in details appears in the commentary of Pṛthudakasvāmī (Colebrooke, *Hindu Algebra*, p. 309 fn.), *Līlāvatī* (p. 38) and *Bījagaṇita* (p. 58) of Bhāskara II. Pṛthudakasvāmī has followed the same method for the solution of this problem as in the case of the previous one. So he differs considerably from others.

[3] *Brāhma-sphuṭa-siddhānta*, xii. 39.

the hill, he descended diagonally to a town. The other walked down the hill and went to the same town. Their journey were equal. I wish to know the distance of the town from the hill, and what is the height of ascent of the super-human ascetic.''

The rule says:

$$\text{Distance of town} = (\text{height of mountain}) \times m$$

$$\text{Height of ascent} = \frac{(\text{height of mountain}) \times m}{m+2}$$

where m is any arbitrary integer.

Problems of this type appear in the *Gaṇita-sāra-saṁgraha*,[1] *Līlāvatī*[2] and *Bījagaṇita*.[3] But while Brahmagupta gives a general solution of his problem considered as an *indeterminate* one, solutions given by Mahāvīra and Bhāskara II are cramped in various ways. Indeed the latters have made the problem *determinate* by certain prescribed limitations. Mahāvīra's rule contemplates a case in which the height of the mountain is two-thirds of the sum of the heights of the mountain and the ascent. Bhāskara supposes the town to be at a given distance from the mountain.

Kaye has made another misstatement about the dependence of Brahmagupta and other Hindu scholars on the Chinese *Chiu-chang Suan-shu*. In enumerating the various topics treated in that work, Kaye writes: "The volume of the cone $= \left(\frac{\text{circumference}}{6}\right)^2 \times$ height,[4] which is given by all the Indians; and the correct volume of a truncated pyramid which is reproduced by Brahmagupta and Śrīdhara.''[5] No part of this statement accurately represents the real facts. It is full of omissions and distortions.

The expression for the volume of a cone occurs in India, so far as known, first in the work of Brahmagupta. He calls a certain kind of solids *sūcī* (literally, "needle") meaning thereby generally a pyramid with a base of any form. The base may be a circle and

[1] vii. 199½-200½.

[2] Colebrooke, *Hindu Algebra*, pp 66-7.

[3] *Ibid*, p. 204.

[4] Inadvertently put as $\left(\frac{\text{circumference}}{6}\right)^3$ in the original.

[5] Kaye, *Indian Math.*, p. 39.

hence the term includes a cone as well. According to Brahmagupta,[1]

$$\text{Volume of cone (or pyramid)} = \frac{(\text{area of base}) \times (\text{height})}{3}$$

This formula reappears in the works of Mahāvīra,[2] Âryabhaṭa II (950),[3] Nemicandra (c. 975),[4] Śrīpati (1039)[5] and Bhāskara II (1150)[6]; but not in the works of Elder Âryabhaṭa (499) and Śrīdhara (c. 750). The formula referred to by Kaye, no doubt, occurs in the works of all these writers[7] except Âryabhaṭa I and Mahāvīra. But it has been specifically noted by all of them that that formula is to be employed only in case of "the measurement of the maunds of grain" (*rāśi-vyavahāra*), not in other cases. It has been further remarked by Brahmagupta, Śrīpati and Bhāskara that in that case the height of the maund must be assumed to be equal to the circumference of the base divided by 9, 10 or 11 according to the kinds of grain. Nemicandra has considered the case (for finer grains) in which the height is one-eleventh of the circumference of the base.[8] So there is absolutely no doubt that the formula was intended only for a "rough calculation," as has been already remarked by the ancient commentators. It appears from an instance given in the *Chang Ch'iu-chien Suan-ching* ("Arithmetical classic of Chang Ch'iu-chien"), belonging probably to the latter half of the sixth century, that Chinese employed their rough formula for the volume of a cone also for the same purpose, *viz.*, the measurement of rice.[9] Professor Mikami does not inform us whether the accurate formula for the volume of a cone was at all known to the ancient Chinese mathematicians.

[1] *Brāhma-sphuṭa-siddhānta*, xii. 44.

[2] Compare *Gaṇita-sāra-saṁgraha*, viii. 17½, 20½.

[3] *Mahāsiddhānta*, xv. 105.

[4] *Triloka-sāra* of Nemicandra, with the commentary of Mādhavacandra (c. 1000), edited by Nathuram Premi, Bombay, 1919, Gāthā 19.

[5] *Siddhānta-śekhara*, xiii. 44.

[6] *Līlāvatī*, p. 65; Colebrooke, *Hindu Algebra*, p. 98.

[7] *Brāhma-sphuṭa-siddhānta*, xii. 50; *Triśatikā*, *R.* 61; *Mahāsiddhānta*, xv. 115; *Triloka-sāra*, Gāthā 22; *Siddhānta-śekhara*, xiii. 50-1, *Līlāvatī*, pp. 69-70.

[8] *Cf.* Gāthā 23 in which has been specified the kinds of seeds for which this assumption is to be made.

[9] Mikami, *Chinese Mathematics*, pp. 42 f.

The formula for the volume of a frustum of a right circular cone appears explicitly in India first in the works of Śrīdhara.[1] If d, D denote the diameters of the upper and lower face of the frustum and h its height, then its volume V will be given by, says Śrīdhara,

$$V = \frac{h}{24}\sqrt{10\{d^2 + D^2 + (d+D)^2\}^2},$$

$$= \frac{\pi}{3}(r^2 + R^2 + rR)h,$$

where r, R denote the radii of the upper and lower face and $\pi = \sqrt{10}$, the value adopted by Śrīdhara. It also reappears in the work of Âryabhaṭa II[2] and Mahāvīra.[3] The accurate formula for the volume of the frustum of a circular cone was not known to the ancient Chinese, though they knew an approximate formula, with $\pi = 3$, from the time of *Chiu-chang Suan-shu*.[4]

Brahmagupta,[5] followed by Mahāvīra,[6] gave a general method of calculating the volume of a frustum of a solid, such as a pyramid, a cone and a wedge, whose upper and lower faces are parallel. He first calculates two approximate values called *Vyavahārika gaṇita* or "approximate volume" (A) and *Autra gaṇita* or "gross volume" (G). Mahāvīra calls them respectively *Karmāntika-phala* and *Auṇḍra-phala*. It is stated that

A = (area from half the sum of the linear dimensions of face and base) × (height)

$G = \frac{1}{2}$ {(area of face) + (area of base)} × (height)

The accurate volume (V) of the frustum is then given to be

$$V = \tfrac{1}{3}(G - A) + A.$$

Now for the frustum of a right circular cone noted before

$$A = \left(\frac{r+R}{2}\right)^2 \times h, \quad G = \tfrac{1}{2}(\pi r^2 + \pi R^2) \times h.$$

[1] *Triśatikā*, Rule 58. This clearly shows that Śrīdhara knew how to find the volume of a complete right circular cone though he had not explicitly recorded it.

[2] *Mahāsiddhānta*, xv. 106.

[3] *Gaṇita-sāra-saṁgraha*, viii. 17½.

[4] Mikami, *Chinese Mathematics*, p. 15; *cf.* also pp. 42 f.

[5] *Brāhma-sphuṭa siddhānta*, xii. 45-6.

[6] *Gaṇita-sāra-saṁgraha*, viii. 9-11½.

$$\therefore \quad V = \frac{\pi}{3}(r^2 + R^2 + rR) \times h.$$

as stated before. Mahāvīra has applied the formula to a few specific examples.[1]

For a truncated wedge-shaped solid, the sides of whose upper face are a, b and the corresponding sides of whose lower face are a', b', the approximate volumes will be

$$A = \left(\frac{a+a'}{2}\right)\left(\frac{b+b'}{2}\right) \times h, \quad G = \tfrac{1}{2}(ab + a'b') \times h.$$

Then the accurate volume of the frustum of the wedge will be

$$V = \frac{1}{3}\left\{\frac{1}{2}\left(ab + a'b'\right)h - \left(\frac{a+a'}{2}\right)\left(\frac{b+b'}{2}\right)h\right\}$$

$$+\left(\frac{a+a'}{2}\right)\left(\frac{b+b'}{2}\right)h. \tag{1}$$

On reduction we easily obtain

$$V = \tfrac{1}{6}\{ab + a'b' + (a+a')(b+b'\} \times h. \tag{2}$$

The formula reappears in this reduced form in the works of Âryabhaṭa II[2], Śrīpati[3] and Bhāskara II.[4]

The formula for the volume of a truncated wedge is given in the *Chiu-chang Suan-shu*[5] and *Mong-hsia Pi-t'an*[6] of Ch'ên Huo (died 1075) as

$$\tfrac{1}{6}\{(2a + a')b + (2a' + a)b'\} \times h.$$

This expression is of course easily reducible to the second Hindu form. But the first Hindu form, the one which has been stated by Brahmagupta and Mahāvīra, is so considerably different from, indeed so more complex than, the Chinese form, it is very difficult to presume that the former has been derived from the latter. The wide divergence in form of the final results cannot be overlooked as

1 *Ibid*, viii. 14½, 17½.
2 *Mahāsiddhānta*, xv. 106.
3 *Siddhānta-śekhara*, xiii. 45.
4 *Līlāvatī*, p. 45.
5 Mikami, *Chinese Mathematics*, p. 15.
6 *Ibid*, p. 61.

immaterial. Indeed in the absence of any other specific evidence the form of a result gives us a clue to the method by which it has been arrived at. The Hindu and Chinese mathematicians seem to have, in fact, followed entirely different methods for calculating the volume of a truncated wedge. Brahmagupta and Mahāvīra obtained their final accurate result through some intermediate stages of rough calculation. The Chinese process seems to have been more direct (*vide infra*).

It should be noted that the above formula for the volume of a truncated wedge is equally available for finding the volume of a truncated pyramid with a rectangular base. Still particularly putting $a=b$, $a'=b'$, we get the formula for the volume of a frustum of a pyramid with a square base as

$$\tfrac{1}{3}(a^2+a'^2+aa')h.$$

This particular formula occurs in the *Chiu-chang Suan-shu*,[1] *Chang Ch'iu-chien Suan-ching*[2] and Heron's *Stereometry*.[3] It was also known to the ancient Egyptians.[4] Cantor,[5] followed by Tropfke[6] and Smith,[7] thinks that Brahmagupta's rule was meant for such a particular case only. But we do not think so.[8] For there is nothing in Brahmagupta's definition of his rule to suggest such a limitation. These writers were probably misled by an example of a well with a square face (a case of a truncated square pyramid turned upside down) given by the commentator Pṛthudakasvāmī in illustration of Brahmagupta's rule.

Mahāvīra's rule is absolutely free from any kind of limitation. For in illustration of it, he has given examples relating to a truncated pyramid with a triangular, square and rectangular base, a frus-

[1] *Ibid*, p. 15.

[2] *Ibid*, pp. 42 f.

[3] T. Heath, *History of Greek Mathematics*, Vol. II, p. 334.

[4] B. Touraeff, *Ancient Egypt*, 1917, pp. 100 ff.

[5] M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Bd. I, Leipzig, 1907, p. 649.

[6] J. Tropfke, *Geschichte der Elementar-Mathematik*, Bd. VII, Leipzig, 1924, pp. 24 f.

[7] D. E. Smith, *History of Mathematics* II, p. 293.

[8] Sudhakara Dvivedi also is of the same opinion. *Vide* his notes on Brahmagupta's rule.

tum of a right circular cone and also a truncated wedge.[1] Now Brahmagupta's rule is substantially the same, not only in the matter of the final result but also in the matter of the definition,—as that of Mahāvīra. So in the absence of any specific facts leading to the contrary, it will be only fair and equitable to believe that Brahmagupta implied the same extent of generality of application by his rule. In any case Kaye is wholly wrong in asserting that Brahmagupta and Śrīdhara have "reproduced" the formula for the volume of a truncated pyramid from the *Chiu-chang Suan-shu.* For no such formula occurs in the *Triśatikā,* the only available work of Śrīdhara.[2] Nor has its occurrence in any of his lost works been testified by any posterior writer. Further such a formula is not found in the *Chiu-chang Suan-shu* or in any other known Chinese treatise on mathematics. If, however, the formula given in the former for the volume of a truncated wedge, be looked upon from that particular point of view, its form, as has already been remarked, is so different from the form of Brahmagupta's formula, one cannot be said to be a reproduction of the other.

To find the volume of a truncated wedge.

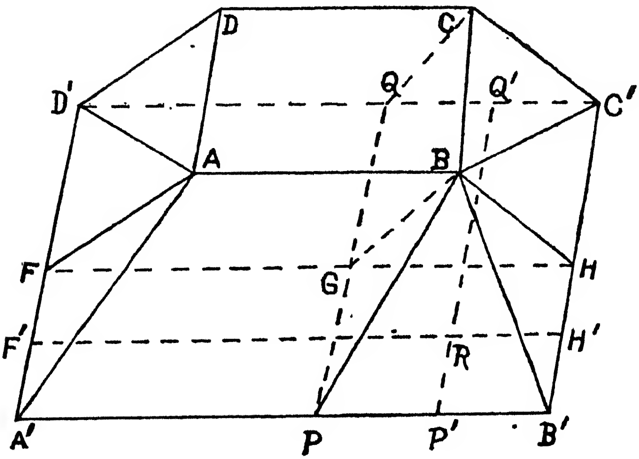
The solid contemplated is a wedge with the rectangular base $A'B'C'D'$ whose upper portion has been cut off by a plane $ABCD$ parallel to the base. Let h be the distance between the two planes and further $AB=DC=a$, $AD=BC=b$, $A'B'=D'C'=a'$, $A'D'=B'C'=b'$.

Method of Chang T'sang: Draw a plane through AB and $C'D'$ so that the solid is divided into two wedges, one with the plane face $A'B'C'D'$ and opposite edge AB and the other with the plane face $ABCD$ and opposite edge $C'D'$. To find the volume of the first wedge, draw two planes through $A'D'$ and $B'C'$ perpendicular to AB (produced if necessary). So that the volume of this wedge is equal to the volume of a triangular prism of base $\frac{1}{2}\,b'h$ and height a'

[1] *Gaṇita-sāra-saṁgraha,* viii. $12\frac{1}{2}$-$18\frac{1}{2}$. There is obviously an error in the example relating to the truncated wedge (viii. $16\frac{1}{2}$). 32 should be 22.

[2] It is perhaps noteworthy that there is no reference to the formula of a truncated pyramid in the English translation of the *Triśatikā* by N. Ramanujacharia which has been published by G. R. Kaye, with his own notes and comments (*Bibl. Math.*, XIII (3), 1912-13, pp. 203-17).

minus the volumes of two prism each having equivalent base and sum of whose heights is $a'-a$, The volume of the first wedge is then[1]



$$\frac{1}{2}\, b'h \times a' - \frac{1}{6} b'h(a'-a),$$

or $$\frac{h}{6}(2a'+a)b'.$$

Similarly the volume of the other wedge is

$$\frac{h}{6}(2a+a')b.$$

Hence the volume of the whole solid is

$$\frac{1}{6}\{(2a+a')b+(2a'+a)b'\}\times h.$$

Q.E.D.

[1] The volume of this wedge can also be obtained in a slightly different way: It is equal to (*i*) the pyramid with the base $PQC'B'$ and vertex B, and (*ii*) the triangular prism with the base $PA'D'Q$ and opposite edge AB. Hence the volume of the wedge is

$$\tfrac{1}{3}(a'-a)b'h+\tfrac{1}{2}ab'h$$

or $$\tfrac{1}{6}(2a'+a)b'h$$

That Chang T'sang did actually follow this method appears very probable from the ease with which the result comes out directly in the form in which it has been stated by him. It is further corroborated by the fact just before stating the formula for the volume of a truncated wedge, Chang T'sang has given the formula for the volume of a complete wedge.

Method of Āryabhaṭa II: Draw a plane through AB parallel to the face $DCC'D'$ cutting $A'D'$ at F and $B'C'$ at H. Similarly through BC draw a plane parallel to the face $ADD'A'$ cutting $A'B'$ at P and $C'D'$ at Q. Join FH and PQ intersecting at G. Then evidently $FD'=b$, $A'P=a$, $A'F=b'-b$, $PB'=a'-a$. Then the given solid is equal to the sum of the parallelopiped $ABCDD'QGF$, wedge $ABGPA'F$, wedge $BCC'HGQ$ and pyramid $BHB'PG$. Now the wedge $ABGPA'F$ is equivalent to half the parallelopiped on the base $A'PGF$ and of height h; the wedge $BCC'HGQ$ is equivalent to half the parallelopiped on the base $C'HGQ$ and of height h; and the pyramid $BHB'PG$ is equivalent to one-third the parallelopiped on the base $HB'PG$ and of height h. Hence the truncated wedge is equivalent to a parallelopiped of height h and base equal to

$$ab+\tfrac{1}{2}a(b'-b)+\tfrac{1}{2}b(a'-a)+\tfrac{1}{3}(b'-b)(a'-a),$$

or $$\tfrac{1}{6}\{ab+a'b'+(a+a')(b+b')\}.$$

Hence the volume of the truncated wedge is

$$\tfrac{1}{6}\{ab+a'b'+(a+a')(b+b')\}\times h.$$

Q.E.D.

Of course this method does not give the final result in the form in which it is recorded as directly as the Chinese method. Nevertheless there are reasons to believe that Āryabhaṭa II and other Hindu writers posterior to him, such as Śrīpati and Bhāskara II did actually follow this method or a method very nearly similar to this. It differs from the Chinese method primarily by the fact that it seeks to find the base of an equivalent parallelopiped of the same height as the given solid. Bhāskara explicitly, others though less so but undoubtedly, refers to the "mean area" of the base of the equivalent parallelopiped.

Method of Brahmagupta: The *rationale* of the rule followed by Brahmagupta, as also by Mahāvīra, for finding the volume of a truncated wedge, cannot be ascertained so easily. For it is a little more complicated than any of the two previous methods. It seems,

however, almost sure that his object was like that of the posterior Hindu writers, to calculate the "mean area" of the base of a parallelopiped whose height as well as the volume will be equal to the corresponding quantities of the given truncated wedge. This he did first roughly by taking the required "mean area" to be equal *once* to the area formed by the mean of the corresponding sides of the face and the base of the given solid, which is also the area of the section of the solid by the mean plane, *i.e.*, the plane intermediate between the two parallel faces of the solid, and *second time* to the mean of the area of the two parallel faces. Thus he obtained two approximate values of the volume of the given solid, *viz.*,

$$G=\tfrac{1}{2}\,(ab+a'b')h,\quad A=\left(\frac{a+a'}{2}\right)\left(\frac{b+b'}{2}\right)h.$$

He then probably calculated the volume accurately by the method indicated above. But still desiring to express the final result in terms of either of those approximate values with necessary corrections, he observed—by drawing the straight lines $F'H'$ bisecting $A'F$, $B'H$ and $P'RQ'$ bisecting PB', QC'—that the "mean base" for the value A, namely $RF'D'Q'$, had fallen short of the "mean base" for the accurate value, by having taken into account only one-fourth of the area $HB'PG$ whereas it ought to have done one-third of that. Hence the correction to A will be the volume of a parallelopiped of height h and of base

$$\frac{1}{12}\,(a'-a)(b'-b).$$

Now this can be expressed as

$$\tfrac{1}{3}\left\{\frac{ab+a'b}{2}-\left(\frac{a'+a}{2}\right)\left(\frac{b'+b}{2}\right)\right\}.$$

So Brahmagupta stated the accurate value of the volume of the truncated wedge as

$$\tfrac{1}{3}\,(G-A)+A.$$

Q.E.D.

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5

A Type of Commutative Matrices.

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1. *Introduction.* If M is a square matrix of order n, any polynomial in M can be reduced by means of the characteristic equation of M to a polynomial of degree $\leqq n-1$ in M, and the general form of a matrix commutative with M is such a polynomial of degree $n-1$.

The type of commutative matrices considered in this note arose in a partial determination of all polynomials in any number of variables that repeat rationally under multiplication in more than one way; they may be of some independent interest.

Each element of a matrix of the type in question is a linear homogeneous function of the same independent variables; if the variables be replaced by any other set, the matrix thus obtained is commutative with the original.

From another, abstractly identical, point of view, the determination of such matrices is equivalent to that of finding n linear homogeneous forms $P_i^{(k)}$,

$$P_i^{(k)} \equiv \sum_{j=1}^{n} c_{ji}^{(k)} x^{(j)}, \quad (k=1, 2, \ldots, n),$$

in the independent variables $x^{(j)}$ such that

$$P_i^{(k)}\left(P_j^{(1)}, P_j^{(2)}, \ldots, P_j^{(n)}\right) = P_j^{(k)}\left(P_i^{(1)}, P_i^{(2)}, \ldots, P_i^{(n)}\right),$$
$$(k=1, 2, \ldots, n).$$

Before proceeding to the general case we give an example, which can be verified, if desired, by actual multiplication of the matrices, although this is unnecessary.

Let p, q, r, s be independent variables, and a, b, c, d arbitrary constants. That the number of variables is equal to that of the constants, is merely a peculiarity of the example, and is of no special significance. For convenience in printing, put

$$p+aq\equiv t, \qquad r+as\equiv u, \qquad p+cr\equiv v,$$

$$q+cs\equiv w, \qquad t+cu\equiv z.$$

Denote the matrix

$$\left\Vert \begin{matrix} p & bq & dr & bds \\ q & t & ds & du \\ r & bs & v & bw \\ s & u & w & z \end{matrix} \right\Vert,$$

considered as a function of p, q, r, s, by $M(p, q, r, s)$. Then, if p', q', r', s' and p'', q'', r'', s'' are any sets of 4 independent variables, $M(p', q', r', s')$ and $M(p'', q'', r'', s'')$ are commutative.

2. *Characters.* A considerable saving of space and labor is affected in the theory of n-dimensional matrices aud multilinear forms by our adaptation of the summation and contraction devices familiar in tensor analysis, which we now explain. In this adaptation superscripts and subscripts do not, as for tensors, indicate contravariant and covariant characters respectively.

Any small Greek letter, $\alpha, \beta, \gamma, \ldots, \xi, \ldots$ denotes a *blank* which may take any one of the values $0, 1, \ldots, n-1$ $(n>1)$. In expressions of the form A_α^β, $B_{\lambda\sigma}^{\gamma\delta\epsilon}$, etc., which will be called *symbols*, $\beta, \gamma, \delta, \epsilon$ are superscripts α, λ, σ subscripts. The same blank can not occur twice as either a subscript or a superscript. A blank may be replaced by any blank not already used in a symbol, without altering the significance of the symbol. All symbols obtained by permuting the subscripts of a given symbol in all possible ways are regarded as the same symbol, and similarly for the superscripts. It is assumed that symbols may be *multiplied*, as $A_\alpha^\beta B_{\lambda\sigma}^{\gamma\delta\epsilon}$; that any *product* of symbols is a unique symbol, and that multiplication of symbols is associative and commutative. It is further assumed that symbols can be added, and that addition is associative and commutative. Lastly, it is assumed that multiplication is distributive over addition.

As in tensors, a repeated blank, for example α in $A^{\alpha}_{\beta\gamma}\, B^{\epsilon}_{\alpha\delta}$, which occurs once as a superscript and once as a subscript, indicates the sum of all symbols obtained from the typical one by putting $\alpha=0, 1,\ldots, n-1$ and adding the results.

The *character* of a symbol or of a product of symbols is the two-rowed array obtained by writing down, in any order, all the superscripts of the symbols in the product to form the upper row, and all the subscripts to form the lower row, and finally deleting any letter which occurs in both the upper and the lower row. Thus the character of $A^{\beta}_{\alpha}\; B^{\gamma\delta\epsilon}_{\lambda\sigma}$ is $\begin{pmatrix} \beta\gamma\delta\epsilon \\ \alpha\lambda\sigma \end{pmatrix}$; that of $A^{\lambda}_{\alpha}\; B^{\gamma\delta\epsilon}_{\lambda\sigma}$ is $\begin{pmatrix} \gamma\delta\epsilon \\ \alpha\sigma \end{pmatrix}$, and the characters obtained from these by permutations of the blanks in the upper row, or in the lower, are identical respectively with the given characters.

An unrepeated blank in a symbol or in a character is called *free.* It is frequently convenient to symbolize a product by an arbitrary single letter with the character of the product; thus $A^{\alpha}_{\beta\lambda}\; B^{\epsilon}_{\alpha\delta} \equiv C^{\epsilon}_{\beta\gamma\delta}$.

A symbol containing precisely s free blanks represents the entire set of n^s symbols obtainable from the given one by letting each of the free blanks range over the integers $0, 1,\ldots, n-1$.

Characters are defined to be *identical* when and only when the numbers of blanks in the upper rows are equal, and likewise for the lower rows.

If to each blank in a character a definite value (one of the integers $0, 1,\ldots, n-1$) be assigned, the result is called a *numerical value* of the character.

Two numerical values of a character are *equal* when and only when they are numerical values of identical characters, and the integers in corresponding places in the two characters are respectively equal.

An equation between symbols has a meaning when and only when their characters are identical; such an equation, when significant, denotes the set of n^s equations obtainable by assigning to the identical characters equal numerical values in all possible ways, s being the number of free blanks in each character. Any assertion concerning a symbol containing precisely s free blanks is equivalent to the simultaneous assertion of the n^s statements corresponding to the n^s numerical values of the blanks.

3. *Operations with matrices.* We need consider only the case of two-dimensional matrices.

Let $\left| A^{\alpha}_{\beta} \right|$ be the square matrix of order n, of which the element in row α and column β is A^{α}_{β}.

If c is a scalar,
$$c \left| A^{\alpha}_{\beta} \right| = \left| cA^{\alpha}_{\beta} \right|.$$

Addition:
$$\left| A^{\alpha}_{\beta} \right| + \left| B^{\delta}_{\gamma} \right| = \left| A^{\alpha}_{\beta} + B^{\alpha}_{\beta} \right|.$$

(1) *Multiplication:*
$$\left| A^{\alpha}_{\beta} \right| \left| B^{\alpha}_{\beta} \right| = \left| A^{\alpha}_{\lambda} \; B^{\lambda}_{\beta} \right|.$$

Transpose: If $A'^{\alpha}_{\beta} \equiv A^{\beta}_{\alpha}$, $\left| A^{\alpha}_{\beta} \right|' = \left| A'^{\alpha}_{\beta} \right|$.

In the product, the character of the typical element $A^{\alpha}_{\lambda} \; B^{\lambda}_{\beta}$ is $\binom{\alpha}{\beta}$, which is also the assignment of the element $A^{\alpha}_{\lambda} \; B^{\lambda}_{\beta}$ to row α, column β in the matric product.

No summation is implied in products such as $\left| A^{\alpha}_{\beta} \right| \left| B^{\alpha}_{\beta} \right|$, as the repeated indices α, β are both upper, or both lower.

Each of the following implies the other:
$$\left| A^{\alpha}_{\beta} \right| = \left| B^{\gamma}_{\delta} \right|, \quad A^{\beta}_{\alpha} = B^{\alpha}_{\beta}.$$

4. *Condition for commutativity.* Consider the two sets x^{α}, y^{β} of n independent variables each and let $c^{\alpha}_{\delta\lambda}$ be n^3 constants (independent of x^{α}, y^{β}). Define X^{α}_{β}, Y^{α}_{β} by
$$x^{\alpha} \, c^{\delta}_{\alpha\xi} \equiv X^{\delta}_{\xi}, \quad y^{\beta} \, c^{\epsilon}_{\beta\delta} \equiv Y^{\epsilon}_{\delta}.$$

Then

$$x^{\alpha} y^{\beta} c^{\delta}_{\alpha\xi} c^{\epsilon}_{\beta\delta} = Y^{\epsilon}_{\delta} X^{\delta}_{\xi},$$

$$x^{\alpha} y^{\beta} c^{\delta}_{\beta\xi} c^{\epsilon}_{\alpha\delta} = X^{\epsilon}_{\delta} Y^{\delta}_{\xi}.$$

Hence

$$\left| x^{\alpha} y^{\beta} c^{\delta}_{\alpha\xi} c^{\epsilon}_{\beta\delta} \right| = \left| Y^{\epsilon}_{\delta} \right| \quad \left| X^{\delta}_{\xi} \right|,$$

$$\left| x^{\alpha} y^{\beta} c^{\delta}_{\beta\xi} c^{\epsilon}_{\alpha\delta} \right| = \left| X^{\epsilon}_{\delta} \right| \quad \left| Y^{\delta}_{\xi} \right|.$$

Hence it follows that, if $\left| X^{\alpha}_{\beta} \right|$, $\left| Y^{\alpha}_{\beta} \right|$ are commutative, it is necessary and sufficient that

$$(2) \quad x^{\alpha} y^{\beta} c^{\delta}_{\alpha\xi} c^{\epsilon}_{\beta\delta} = x^{\alpha} y^{\beta} c^{\delta}_{\beta\xi} c^{\epsilon}_{\alpha\delta}.$$

In (2) there are precisely 4 free blanks. Hence (2) is a system of n^4 equations. Now x^{α}, y^{β} are $2n$ independent variables. Thus (2) is equivalent to a set of $n^4 - 2n$ conditions upon the n^3 constants $c^{\alpha}_{\beta\gamma}$. In order that $\left| X^{\alpha}_{\beta} \right|$, $\left| Y^{\alpha}_{\beta} \right|$ shall be commutative, it is necessary and sufficient that the indicated conditions be satisfied.

Having determined $c^{\alpha}_{\beta\gamma}$ so that (2) is satisfied, we shall have found a matrix $\left| X^{\alpha}_{\beta} \right|$, whose elements are linear homogeneous functions of the n independent variables x^{α}, such that, if $\left| Y^{\alpha}_{\beta} \right|$ is the matrix obtained from $\left| X^{\alpha}_{\beta} \right|$ by replacing x^{α} by y^{α}, the matrix $\left| Y^{\alpha}_{\beta} \right|$ is commutative with $\left| X^{\alpha}_{\beta} \right|$.

The element in row α, column β of $\left| X^{\alpha}_{\beta} \right|$ is $x^{\delta} c^{\alpha}_{\delta\beta}$.

5. *Determination of the constants.* Let f_{α} be n linearly independent elements of a ring, namely, of a system closed under addition, multiplication and subtraction, and assume for a moment that constants $c^{\alpha}_{\beta\lambda}$ exist such that

$$f_{\alpha} f_{\beta} = f_{\beta} f_{\alpha} = c^{\delta}_{\alpha\beta} f_{\delta} . \tag{3}$$

Then if u^{ξ} is an independent variable,

$$x^{\alpha} f_{\alpha} u^{\xi} f_{\xi} = x^{\alpha} u^{\xi} f_{\alpha} f_{\xi} = x^{\alpha} u^{\xi} c^{\delta}_{\alpha\xi} f_{\delta} ,$$

$$y^{\beta} f_{\beta} u^{\xi} f_{\xi} = y^{\beta} u^{\xi} f_{\beta} f_{\xi} = y^{\beta} u^{\xi} c^{\delta}_{\beta\xi} f_{\delta} .$$

Hence

$$y^{\beta} f_{\beta} x^{\alpha} f_{\alpha} u^{\xi} f_{\xi} = x^{\alpha} y^{\beta} u^{\xi} c^{\delta}_{\alpha\xi} c^{\epsilon}_{\delta\beta} f_{\epsilon} ,$$

$$x^{\alpha} f_{\alpha} y^{\beta} f_{\beta} u^{\xi} f_{\xi} = x^{\alpha} y^{\beta} u^{\xi} c^{\delta}_{\beta\xi} c^{\epsilon}_{\delta\alpha} f_{\epsilon} ,$$

the first of which is merely the expression for the product of the three linear forms

$$y^{\alpha} f_{\alpha} , \; x^{\alpha} f_{\alpha} , \; u^{\alpha} f_{\alpha} ,$$

the multiplications being performed in the order written from right to left, and similarly for the second and

$$x^{\alpha} f_{\alpha} , \; y^{\alpha} f_{\alpha} , \; u^{\alpha} f_{\alpha} .$$

Since multiplication is commutative,

$$x^{\alpha} y^{\beta} u^{\xi} c^{\delta}_{\alpha\xi} c^{\epsilon}_{\delta\beta} f_{\epsilon} = x^{\alpha} y^{\beta} u^{\xi} c^{\delta}_{\beta\xi} c^{\epsilon}_{\delta\alpha} f_{\epsilon} .$$

Since the f_α are linearly independent, it follows that

$$x^\alpha y^\beta u^\xi c^\delta_{\alpha\xi} c^\epsilon_{\delta\beta} = x^\alpha y^\beta u^\xi c^\delta_{\beta\xi} c^\epsilon_{\delta\alpha},$$

in which ϵ is free. Since $f_\alpha f_\beta = f_\beta f_\alpha$, $c^\mu_{\alpha\beta} = c^\mu_{\beta\alpha}$. Hence

$$x^\alpha y^\beta u^\xi c^\delta_{\alpha\xi} c^\epsilon_{\beta\delta} = x^\alpha y^\beta u^\xi c^\delta_{\beta\xi} c^\epsilon_{\alpha\delta},$$

and therefore since u^ξ represents n independent variables,

$$x^\alpha y^\beta c^\delta_{\alpha\xi} c^\epsilon_{\beta\delta} = x^\alpha y^\beta c^\delta_{\beta\xi} c^\epsilon_{\alpha\delta},$$

which is the requisite condition (2) of the preceding section.

If remains therefore only to find sets of n Indeterminates f_α satisfying (3).

Consider the general equation of degree $n>1$ in f,

(4) $$f^n + t_1 f^{n-1} + \cdots + t_n = 0,$$

where $t_1, \cdots, t_n$ are independent variables. Write $f_\alpha \equiv f^\alpha$. Then any polynomial in f is uniquely reducible by means of (4) to a polynomial in f of degree $\leqq n-1$. In particular the product $f_\alpha f_\beta$ is so reducible. Hence the $c^\delta_{\alpha\beta}$ are uniquely defined by $f_\alpha f_\beta = c^\delta_{\alpha\beta} f_\delta$, and we have

$$c^\delta_{\alpha\beta} = c^\delta_{\beta\alpha}.$$

Instead of a single general equation (4) of degree $n>1$ in f we may proceed similarly with s general equations of degrees $n_1, n_2, \ldots, n_s$ in $f, g, \ldots, h$ respectively. The typical product is then $f_\alpha g_\beta \ldots h_\delta f_\lambda g_\mu \ldots h_\nu$, instead of $f_\alpha f_\beta$ as before, where α, λ range from 0 to n_1-1; β, μ from 0 to n_2-1;...; δ, ν from 0 to n_s-1. The example in § 1 was thus constructed from the case $s=2$, $n_1=n_2=2$.

Since the general equation is irreducible, it follows in an obvious manner that the matrices constructed as outlined are non singular, and hence have unique inverses. They therefore generate an abelian group.

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1

On the zeros of Non-differentiable Functions of Darboux's type.

BY

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The object of the present paper is to investigate the zeros of the non-differentiable functions of Darboux's type, namely, those non-differentiable functions, the simplest of which are the functions

$$D(x) \equiv \sum_{1}^{\infty} \frac{6^n \cos\{1.3.5...(2n-1)\}\pi x}{1.3.5......(2n-1)}$$

and
$$\overline{D}(x) \equiv \sum_{1}^{\infty} \frac{6^n \sin\{1.5.9...(4n+1)\}\pi x}{1.5.9.....(4n+1)},$$

given * by Darboux in 1879 and the most general are the functions given † by Lerch in 1888, *e.g.*, the function

$$L(x) \equiv \sum_{1}^{\infty} \frac{r^n \cos(a_n \pi x)}{a_n}$$

where $r>1$, $a_n = p_0 \, p_1 \, p_2 ... p_n$, the p's being any odd integers with p_n tending to infinity as n tends to infinity.

It is believed that those, who wished to give "graphical representations" of non-differentiable functions, as Wiener, Felix Klein and G. C. Young ‡ did in the case of Weierstrass's function, must have

* *Ann. de l'E'cole Normale* (2), VIII, pp. 195-202.

† *Crelle's Journal*, Bd. 103, pp 135-138. It should be noted that. like Cellerier's function, Lerch's function may have infinite (but nowhere finite) differential coefficients.

‡ Wiener, "Ueber die Weierstrasssche Function" (*Crelle's Journal*, Bd, 90, 1881) F. Klein, *Anwendungen der Differentialrechnung und Integralrechnung auf die Geometrie*, 1907; G. C. Young, "On infinite derivates," (*Quarterly Journal of Mathematics*, Vol. 47, 1916, see specially p. 156).

desired to know the zeros of those functions. However, it is only recently that the first successful investigation of this kind has been published by Professor Ganesh Prasad * who has given general expressions from which zeros of Weierstrass's function

$$\sum_{0}^{\infty} a^n \cos (b^n \pi x)$$

can be obtained. Professor Prasad has also suggested the following † problem: "To classify non-differentiable functions according to the number of limiting points which its zeros possess in a finite interval."

The investigation, the results of which are embodied in the present paper, was undertaken at the suggestion of Professor Prasad to whom I wish to express my sincere thanks for his interest and encouragement.

In § 1, zeros of

$$D(x) \equiv \sum_{1}^{\infty} \frac{6^n \cos \{1.3\ldots.(2n-1)\} \pi x}{1.3\ldots.(2n-1)}$$

have been investigated, and it has been proved that, whatever odd positive integer k may be, there is a zero between

$$\frac{1}{2} + \frac{1}{3.5\ldots.(2k+1)} \text{ and } \frac{1}{2} + \frac{1}{3.5.\ \ (2k+3)};$$

and the general expression for zeros other than $\frac{1}{2}$ in the interval (0,1) is given as

$$\frac{1}{2} \pm \frac{1}{3.5\ldots.(2k+1)}(1-\lambda_k)$$

where $0<\lambda_k<1$ and λ_k can be approximated to as closely as we please. By way of illustration it has been shewn that

$$\cdot 11486 < \lambda_1 < \cdot 11488,$$

$$\cdot 22 < \lambda_3 < \cdot 21.$$

* "On the zeros of Weierstrass's non-differentiable function." (*Proceedings of the Benares Mathematical Society*, Vol. XI.)

† *l. c.*, p. 8,

In § 2 zeros of

$$\bar{D}(x) = \frac{6^n \sin\{1.5....(4n+1)\}\pi x}{1.5....(4n+1)}$$

have been studied and the general expression for zeros other than the origin and the points $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, in the interval (0, 1) is given by

$$\frac{1}{5.9....(4l+1)}\left(1+\frac{1}{2}\mu_l\right)$$

where l is any positive integer and μ_l is such that $0 < \mu_l < 1$ and μ_l like λ_k may be approximated to as closely as one wishes. This section finishes with a formula for approximation in the general case.

In § 3, zeros of Lerch's function

$$L(x) \equiv \sum_{n=0}^{\infty} \frac{r^n \cos(a_n \pi x)}{a_n}$$

have been investigated.

The first portion deals with the classification of the function into three classes according to the nature of the positive odd integers the p's; and the last contains the detailed discussion of the class A in which the p's are all of the form $4m+3$, where m may be zero or any positive integer. It is proved here, that there is a zero of $L(x)$ of class A between

$$\frac{1}{2}+\frac{3/2}{p_0 \cdots p_{2t+1}} \text{ and } \frac{1}{2}+\frac{3/2}{p_0 \cdots p_{2t+2}},$$

for every positive integral value of t including zero.

The paper concludes with a brief discussion of the other two classes.

§ 1

Zeros of Darboux's cosine function $D(x)$

1. *There is a zero of $D(x)$ between* $\frac{1}{2}+\frac{1}{3}$ and $\frac{1}{2}+\frac{1}{3.5}$.

Proof:—

$$D(x) = \frac{6}{1}\cos(\pi x) + \frac{6^2}{1.3}\cos(1.3\pi x) + \frac{6^3}{1.3.5}\cos(1.3.5\,\pi x) + \ldots$$

$$D\left(\frac{1}{2}+\frac{1}{3}\right)=\frac{6}{1}\left\{-\sin\frac{\pi}{3}\right\}+0+0+\ldots\ldots<0$$

and

$$D\left(\frac{1}{2}+\frac{1}{3.5}\right)=\frac{6}{1}\left\{-\sin\frac{\pi}{3\cdot5}\right\}+\frac{6^2}{1.3}\sin\frac{\pi}{5}+0+0+\ldots \qquad (1)$$

Now we have the well known trigonometric relation

$$1>\frac{\sin\theta}{\theta}>\frac{2}{\pi} \text{ for } 0<\theta<\frac{\pi}{2}.$$

In order to determine the sign of

$$D\left(\frac{1}{2}+\frac{1}{3.5}\right),$$

we first increase the absolute value of the negative term, *i.e.*, the first term in $D\left(\frac{1}{2}+\frac{1}{3.5}\right)$, and diminish the positive term, *i.e.*, the second term in $D\left(\frac{1}{2}+\frac{1}{3,5}\right)$.

To effect this, we substitute in (1)

$$\frac{\pi}{15} \text{ for } \sin\frac{\pi}{15} \text{ and } \frac{2}{5} \text{ for } \sin\frac{\pi}{5}.$$

Thus

$$D\left(\frac{1}{2}+\frac{1}{3.5}\right)>-\frac{6}{1}\cdot\frac{\pi}{15}+\frac{6^2}{1.3}\cdot\frac{2}{5}>0,$$

It is useless to remark that $D\left(\frac{1}{2}+\frac{1}{1.3.5}\right)$ will still more be positve if we consider the actual values of the sines. Thus, remembering that $D(x)$ is continuous, it takes up all the values lying between

$D\left(\frac{1}{2}+\frac{1}{3}\right)$ and $D\left(\frac{1}{2}+\frac{1}{3.5}\right)$; so there must be at least one value of x between $\frac{1}{2}+\frac{1}{3}$ and $\frac{1}{2}+\frac{1}{3.5}$, for which $D(x)$ vanishes.

2. *There is a zero of $D(x)$ between*

$$\frac{1}{2}+\frac{1}{3.5.7} \text{ and } \frac{1}{2}+\frac{1}{3.5.7.9}$$

Proof :—

$$D\left(\frac{1}{2}+\frac{1}{3.5.7}\right)=\frac{6}{1}\left\{-\sin\frac{\pi}{3.5.7}\right\}+\frac{6^2}{1.3}\sin\frac{\pi}{5.7}$$

$$+\frac{6^3}{1.3.5}\sin\frac{\pi}{7}+0+0+\ldots$$

$$>-\frac{6\pi}{3.5.7}+\frac{6^2}{1.3}\cdot\frac{2}{5.7}+\frac{6^3}{1.3.5}\ \frac{2}{7}>0.$$

Also

$$D\left(\frac{1}{2}+\frac{1}{3.5.7.9}\right)=\frac{6}{1}\left\{-\sin\frac{\pi}{3.5.7.9}\right\}+\frac{6^2}{1.3}\sin\frac{\pi}{5.7.9}$$

$$+\frac{6^3}{1.3.5}\sin\frac{\pi}{7.9}-\frac{6^4}{1.3.5.7}\sin\frac{\pi}{9}+0+\ldots$$

To determine its sign, we follow a principle similar to that in Art 1 ; we diminish the absolute value of each negative term and increase each positive term.

Thus

$$D\left(\frac{1}{2}+\frac{1}{3.5.7.9}\right)$$

$$<-\frac{6}{1}\cdot\frac{2}{3.5.7.9}+\frac{6^2}{1.3}\cdot\frac{\pi}{5.7.9}+\frac{6^3\pi}{1.3.5.7.9}-\frac{6^4.2}{1.3.5.7.9}$$

$$<0.$$

Therefore, there is a root of $D(x)=0$ between

$$\frac{1}{2}+\frac{1}{3.5.7} \text{ and } \frac{1}{2}+\frac{1}{3.5.7.9}.$$

3. *Generally, there is a zero of $D(x)$ between*

$$\frac{1}{2}+\frac{1}{3.5\ldots\ldots(2k+1)} \text{ and } \frac{1}{2}+\frac{1}{3.5\ldots\ldots(2k+1)\,(2k+3)},$$

where k may be any odd integer.

For proving the above result one should bear in mind the remarks under (*a*), (*b*), (*c*) and (*d*) below :—

(*a*) In order to understand clearly the general case, it is desirable to notice at the outset the following cyclic changes of signs of the cosines in the terms of the series for $D(x)$:

$$\text{Cos}\left\{1.\frac{\pi}{2}+\theta\right\}=-\sin\theta, \quad \text{Cos}\left\{1.3.5.7.9.\frac{\pi}{2}+\theta\right\}=-\sin\theta,$$

$$\text{Cos}\left\{1.3.\frac{\pi}{2}+\theta\right\}=+\sin\theta, \quad \text{Cos}\left\{1.3.\ .9.11.\frac{\pi}{2}+\theta\right\}=+\sin\theta,$$

$$\text{Cos}\left\{1.3.5.\frac{\pi}{2}+\ \right\}=+\sin\theta, \quad \text{Cos}\left\{1.3\ldots11.13.\frac{\pi}{2}+\theta\right\}=+\sin\theta,$$

$$\text{Cos}\left\{1.3.5.7.\frac{\pi}{2}+\theta\right\}=-\sin\theta, \quad \text{Cos}\left\{1.3\ldots13.15.\frac{\pi}{2}+\theta\right\}=-\sin\theta,$$

and so on.

(*b*) $$D\left(\frac{1}{2}+\frac{1}{3.5\ldots(2k+1)}\right)$$

$$=\frac{6}{1}\left\{-\sin\frac{\pi}{3.5\ldots(2k+1)}\right\}+\frac{6^2}{1.3}\sin\frac{\pi}{5.7\ldots(2k+1)}+\ldots$$

$$\pm\frac{6^{k-1}}{1.3\ldots(2k-3)}\sin\frac{\pi}{(2k-1)\,(2k+1)}\pm\frac{6^k}{1.3.5\ldots(2k-1)}\sin\frac{\pi}{2k+1}+0\ldots$$

and

$$D\left(\frac{1}{2}+\frac{1}{3.5\ldots(2k+1)\,(2k+3)}\right)$$

$$=\frac{6}{1}\left\{-\sin\frac{\pi}{3.5\ldots(2k+3)}\right\}+\frac{6^2}{1.3}\sin\frac{\pi}{5.7\ldots(2k+3)}+\ldots$$

$$\pm \frac{6^k}{1.3.5\ldots(2k-1)} \sin \frac{\pi}{(2k+1)(2k+3)}$$

$$\pm \frac{6^{k+1}}{1.3\ldots(2k+1)} \sin \frac{\pi}{(2k+3)} + 0 + \ldots$$

In order that there may be a root of $D(x) = 0$ between

$$\frac{1}{2} + \frac{1}{3.5\ldots(2k+1)} \text{ and } \frac{1}{2} + \frac{1}{3.5\ldots(2k+3)};$$

it is sufficient that

$$D\left(\frac{1}{2} + \frac{1}{3.5\ldots(2k+3)}\right) \text{ and } D\left(\frac{1}{2} + \frac{1}{3.5\ldots(2k+3)}\right)$$

be of different signs.

(c) *Case I*:

$$D\left(\frac{1}{2} + \frac{1}{3.5\ldots(2k+1)}\right) > 0,$$

and

$$D\left(\frac{1}{2} + \frac{1}{3.5\ldots(2k+3)}\right) < 0.$$

It will be shewn presently that $D(x)$ is positive or negative, according as the last term in its expansion is positive or negative.

Therefore

$$D\left(\frac{1}{2} + \frac{1}{3.5\ldots(2k+1)}\right)$$

is positive if its last term, *i.e.*,

$$\pm \frac{6^k}{1.3\ldots(2k-1)} \cdot \sin \frac{\pi}{(2k+1)} \quad \text{is so, } i.e.,$$

$$\left\{1.3\ldots\overline{2k-1}\right\}\frac{\pi}{2} = 2\pi m_1 + \frac{3\pi}{2}$$

and similarly

$$D\left(\frac{1}{2} + \frac{1}{3.5\ldots(2k+3)}\right)$$

is negative if the last term in its expansion, *i.e.*,

$$\pm \frac{6^{k+1}}{1.3\ldots(2k+1)} \cdot \sin \frac{\pi}{(2k+3)} \text{ is so, } i.e.,$$

$$\left\{1.3\ldots \overline{(2k+1)}\right\} \frac{\pi}{2} = 2\pi m_2 + \frac{\pi}{2}$$

where m_1 and m_2 are positive integers. For the fulfilment of the above conditions, the k^{th} term, *i. e.*, $(2k\text{-}1)$ of the sequence 1, 3, 5...must be restricted to be of the form $8p-3$ for some positive integral value of p.

(*d*) *Case II* :

There will also be a root of $D(x) = 0$ between

$$\frac{1}{2} + \frac{1}{3.5\ldots(2k+1)} \text{ and } \frac{1}{2} + \frac{1}{3.5\ldots(2k+3)},$$

$$\text{if } D\left(\frac{1}{2} + \frac{1}{3.5\ldots(2k+1)}\right) < 0$$

$$\text{and } D\left(\frac{1}{2} + \frac{1}{3.5\ldots(2k+3)}\right) > 0;$$

i.e., in this case

$$\left\{1.3.5\ldots(2k-1)\right\} \frac{\pi}{2} = 2\pi n_1 + \frac{\pi}{2}$$

$$\text{and } \left\{1.3.5\ldots(2k+3)\right\} \frac{\pi}{2} = 2\pi n_2 + \frac{3\pi}{2}$$

where, n_1 and n_2 are any positive integral numbers.

In this case, the k^{th} term of the same sequence 1, 3, 5,...must be of the form $8p-7$ for some positive integral value of p.

4. We proceed to give the proof of the statement in the beginning of Art. 3, Case I.

Proof:—

$$(a)\ D\left(\frac{1}{2} + \frac{1}{3.5\ldots(2k+1)}\right)$$

$$= \frac{6}{1} \left\{ - \sin \frac{\pi}{3.5 \ldots (2k+1)} \right\} + \ldots$$

$$\pm \frac{6^{k-1}}{1.3 \ldots (2k-3)} \sin \frac{\pi}{(2k-1)(2k+1)} + \frac{6^k}{1.3 \ldots (2k-1)} \sin \frac{\pi}{(2k+1)} .$$

Let us examine the most unfavourable case, *i.e.*, let us suppose that all the terms excluding the last, are negative in

$$D\left(\frac{1}{2} + \frac{1}{3.5 \ldots (2k+1)} \right);$$

then we adopt the same principle as before, namely, here we increase the absolute value of every term thus supposed negative and diminish the last positive term and take the difference.

Thus

$$D\left(\frac{1}{2} + \frac{1}{3.5 \ldots . (2k+1)} \right)$$

$$> \frac{2.6^k}{1.3 \ldots . (2k+1)} - \frac{\pi . 6}{1.3 . \ldots (2k+1)} . \left\{ \frac{6^{k-1} - 1}{5} \right\}$$

$$= \frac{1}{1.3 \ldots . (2k+1)} \left\{ 2.6^k - \frac{\pi (6^k - 6)}{5} \right\}$$

$$= \frac{1}{1.3 \ldots . (2k+1)} \left\{ \frac{6^k (10 - \pi) + 6\pi}{5} \right\} > 0$$

(for $10 > \pi$).

Therefore it is found that

$$D\left(\frac{1}{2} + \frac{1}{3.5 \ldots . (2k+1)} \right) > 0.$$

(*b*) $D\left(\frac{1}{2} + \frac{1}{3.5 \ldots . (2k+3)} \right)$

$$= \frac{6}{1} \left\{ - \sin \frac{\pi}{3.5 \ldots . (2k+3)} \right\} + \ldots \ldots$$

$$\pm \frac{6^k}{1.3....(2k-1)}\left\{ \sin \frac{\pi}{(2k+1)(2k+3)} \right\} - \frac{6^{k+1}}{1.3....(2k+1)} \sin \frac{\pi}{(2k+3)} .$$

Proceeding exactly as in (a) above, we find that

$$D\left(\frac{1}{2} + \frac{1}{3.5....(2k+3)}\right)$$

$$< \frac{\pi.6}{1.3....(2k+3)}\left\{ 1 + 6 + ... + 6^{k-1} \right\} - \frac{1 \quad 2.6^{k+1}}{1.3....(2k+3)}$$

$$= \frac{6\ \pi}{1.3....(2k+3)}\left\{ \frac{6^k-1}{5} \right\} - \frac{2.6^{k+1}}{1.3....(2k+3)}$$

$$= \frac{1}{1.3....(2k+3)}\left\{ \frac{6^{k+1}(\pi-10)-6\ \pi}{5} \right\} < 0.$$

Therefore there is a root between

$$\frac{1}{2} + \frac{1}{3.5....(2k+1)} \text{ and } \frac{1}{2} + \frac{1}{3.5....(2k+3)}.$$

5. *Case II.*

Proceeding exactly as in *Case I*, we come to the same conclusion.

Also, it can be proved that there is a root of $D(x)=0$ between

$$\frac{1}{2} - \frac{1}{3} \text{ and } \frac{1}{2} - \frac{1}{3.5};$$

and, generally, there is a root between

$$\frac{1}{2} - \frac{1}{3.5....(2k+1)} \text{ and } \frac{1}{2} - \frac{1}{3.5....(2k+3)}.$$

6. *Every zero can be approximated to as closely as we please.*

Thus, one-fold infinity of zeros of $D(x)$ between $(0, 1)$ are

$$\frac{1}{2}, \frac{1}{2} \pm \frac{1}{3.5....(2k+1)}(1-\lambda_k)$$

k being any odd integer and λ_k is a number lying between 0 and 1,

which can be approximated to as closely as we please, as illustrated below for

$$k=1, 3.$$

By direct calculation, it is found that

(a) $D\left(\frac{1}{2}+\frac{1}{3}\right) < 0, \qquad D\left(\frac{1}{2}+\frac{1}{15}\right) > 0,$

$$D\left(\frac{1}{2}+\frac{4}{15}\right) > 0, \qquad D\left(\frac{1}{2}+\frac{34}{105}\right) > 0,$$

$$D\left(\frac{1}{2}+\frac{306}{945}\right) > 0, \qquad D\left(\frac{1}{2}+\frac{307}{945}\right) < 0,$$

$$D\left(\frac{1}{2}+\frac{3066}{10395}\right) > 0, \qquad D\left(\frac{1}{2}+\frac{3367}{10395}\right) < 0,$$

$$D\left(\frac{1}{2}+\frac{39870}{135135}\right) > 0, \qquad D\left(\frac{1}{2}+\frac{39871}{135135}\right) < 0.$$

Now, if

$$D\left\{\frac{1}{2}+\frac{1}{3}(1-\lambda_1)\right\}=0,$$

then we find that $\cdot 11486 < \lambda_1 < \cdot 11488$.

By further calculation, λ_1 may be approximated to more closely.

(b) When $k=3$,

$$D\left(\frac{1}{2}+\frac{1}{3.5.7}\right) > 0, \qquad D\left(\frac{1}{2}+\frac{1}{3.5.7.9}\right) < 0,$$

$$D\left(\frac{1}{2}+\frac{7}{945}\right) < 0, \qquad D\left(\frac{1}{2}+\frac{8}{945}\right) > 0,$$

$$D\left(\frac{1}{2}+\frac{77}{10395}\right) < 0, \qquad D\left(\frac{1}{2}+\frac{78}{10395}\right) > 0.$$

so if

$$D\left\{\frac{1}{2}+\frac{1}{3.5.7}(1-\lambda_3)\right\}=0,$$

whence we find, the value of λ_3 lying between

$$\cdot 22 > \lambda_3 > \cdot 21$$

§ 2

Zeros of Darboux's Sine Function : $\bar{D}(x)$

7. There is a zero of $\bar{D}(x)$ between

$$\frac{1}{5\cdot 9} \text{ and } \frac{3/2}{5\cdot 9}.$$

Proof :—

$$\bar{D}\left(\frac{1}{5.9}\right)=\frac{6}{5}\cdot \sin\frac{\pi}{9} > 0.$$

Also

$$\bar{D}\left(\frac{3/2}{5.9}\right)=\frac{3}{5}-\frac{6^2}{5.9}-\frac{6^2}{5.9.13}-\cdots\cdots$$

$$=\frac{3}{5}-\frac{4}{5}-\cdots\cdots$$

$$< 0.$$

Therefore there is at least one root of

$$\bar{D}(x)=0 \text{ between } \frac{1}{5.9} \text{ and } \frac{3/2}{5.9}.$$

8. There is a zero of $\bar{D}(x)$ between $\dfrac{1}{5.9.13}$ and $\dfrac{3/2}{5.9.13}$.

Proof :—

$$\bar{D}\left(\frac{1}{5.9.13}\right)=\frac{6}{5}\sin\frac{\pi}{9.13}+\frac{6^2}{5.9}\sin\frac{\pi}{13}+0+\cdots\cdots$$

$$>0.$$

$$\bar{D}\left(\frac{3/2}{5.9.13}\right)=\frac{6}{5}\cdot\sin\frac{3\pi/2}{9.13}+\frac{6^2}{5.9}\sin\frac{3\pi/2}{13}-\frac{6^3}{5.9.13}-\cdots\cdots$$

$$<\frac{3\pi/2}{5.9.13}\cdot\left\{6+6^2\right\}-\frac{6^3}{5.9.13}-\frac{6^4}{5.13.17}-\cdots\cdots$$

$$< \frac{1}{5.9.13}\left\{42.\frac{3\pi}{2}-6^3\right\}-\frac{6^4}{5.9.13.17}-\ldots\ldots$$

$$< \frac{1}{5.9.13}\left\{201\cdot 6-216\right\}-\ldots\ldots\ldots$$

$$< \frac{-14.4}{5.9.13}$$

$$< 0.$$

Hence it is proved that there is a root of

$$\bar{D}(x)=0 \text{ between } \frac{1}{5.9.13} \text{ and } \frac{3/2}{5.9.13}.$$

9. *Generally, there is a zero between*

$$\frac{1}{5.9.\ldots.(4l+1)} \text{ and } \frac{3/2}{5.9.\ldots(4l+1)},$$

where l may be any positive integer.

Proof:—

(a) $\bar{D}\left(\frac{1}{5.9\ldots(4l+1)}\right)$

$$=\frac{6}{5}.\sin\frac{\pi}{9.13.\ldots(4l+1)}+\frac{6^2}{5.9}\sin\frac{\pi}{13.17.\ldots(4l+1)}+\ldots$$

$$+\frac{6^{l-1}}{5.9.\ldots(4l-3)}\sin\frac{\pi}{4l+1}+0+0+\ldots$$

$$> 0.$$

(b) $\bar{D}\left(\frac{3/2}{5.9.\ldots(4l+1)}\right)$

$$=\frac{6}{5}.\sin\frac{3\pi/2}{9.13.\ldots(4l+1)}+\frac{6^2}{5.9}\sin\frac{3\pi/2}{13.17.\ldots(4l+1)}+\ldots$$

$$+\ldots+\frac{6^{l-1}}{5.9.\ldots(4l-3)}\sin\frac{3\pi/2}{(4l+1)}-\frac{6^l}{5.9.\ldots(4l+1)}-\ldots$$

Here, also, in order to determine the sign of $\overline{D}\left(\frac{3/2}{5.9\ldots.(4l+1)}\right)$, we adopt the same process of evaluation as before.

Thus $\overline{D}\left(\frac{3/2}{5.9\ldots.(4l+1)}\right)$

$$< \frac{6}{5}\cdot\frac{3\pi/2}{9\ldots.(4l+1)}+\frac{6^2}{5.9}\cdot\frac{3\pi/2}{13.17\ldots.(4l+1)}+\ldots$$

$$+\frac{6^{l-1}.3\pi/2}{5.9\ldots.(4l+1)}-\frac{6^l}{5.9\ldots.(4l+1)}-\ldots$$

$$=\frac{(3\pi/2).6\{1+6+6^2+\ldots+6^{l-2}\}-6^l}{5.9\ldots.(4l+1)}$$

or

$$\frac{1}{5.9\ldots.(4l+1)}\left\{\frac{6^l(3\pi/2-5)}{5}\right\}-\frac{3\pi/2}{5.5.9\ldots.(4l+1)}-\ldots$$

or

$$\frac{1}{5.9\ldots.(4l+1)}\left\{\frac{6^l(3\pi-10)}{10}\right\}-\frac{3\pi/2}{10.5.9\ldots(4l+1)}-\ldots<0$$

for $3\pi < 9\cdot 45 < 10$.

Thus, it is proved that there must be at least one root between

$$\frac{1}{5.9\ldots.(4l+1)} \text{ and } \frac{3/2}{5.9\ldots.(4l+1)}.$$

If we consider the interval $(-1,1)$, it may be easily seen that there is a zero of $\overline{D}(x)$ between

$$-\frac{1}{5.9\ldots.(4l+1)} \text{ and } -\frac{3/2}{5.9\ldots.(4l+1)}$$

10. *Zeros of* $\overline{D}(x)$ *between* $(0, 1)$ *are*

$$0, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \frac{1}{5.9\ldots.(4l+1)}\left(1+\frac{1}{2\mu_l}\right)$$

where l is any positive integer and μ_l is a number lying between 0 *and* **1** *which can be approximated to as closely as we please.*

(a) As illustrations, consider,

$$\bar{D}\left\{\frac{1}{5.9}\left(1+\frac{1}{2}\mu_2\right)\right\}=0.$$

Let $\bar{D}\left\{\frac{1}{5.9}+\frac{m}{5.9.13}\right\}=0,$

where m is an integer.

Therefore we have

$$\frac{6}{5}\sin\left\{\frac{\pi}{9}+\frac{m\pi}{9.13}\right\}-\frac{6^2}{5.9}\cdot\sin\frac{m\pi}{13}=0\,;$$

whence $m=2$.

Next suppose

$$\bar{D}\left\{\frac{1}{5.9}+\frac{2}{5.9.13}+\frac{k}{5.9.13.17}\right\}=0,$$

where k is any integer.

So that we have by substitution in $\bar{D}(x)=0$, *i.e.*,

$$\frac{6}{5}\cdot\sin\left(\frac{\pi}{9}+\frac{2\pi}{9.13}+\frac{k\pi}{9.13.17}\right)-\frac{6^2}{5.9}\sin\left(\frac{2\pi}{13}+\frac{k\pi}{13.17}\right)$$

$$-\frac{6^3}{5.9.13}\cdot\sin\frac{k\pi}{17}=0$$

whence $k=1$.

Thus, up to the second approximation

$$\mu_2=\frac{4}{13}+\frac{2}{13.17}\,.$$

Hence, it follows, from above, that μ_2 can be approximated to more and more closely.

(b) Let $\frac{1}{5.9....(4l+1)}+\frac{m}{5.9,....(4l+5)}$ be a root of $\bar{D}(x)=0$.

Hence, we have by substitution in $\overline{D}(x)=0$

$$\frac{6}{5}\cdot\sin\left(\frac{\pi}{9\ldots.(4l+1)}+\frac{m\pi}{9\ldots.(4l+5)}\right)+\ldots$$

$$+\frac{6^{l-1}}{5.9\ldots.(4l-3)}\sin\left(\frac{\pi}{(4l+1)}+\frac{m\pi}{(4l+1)(4l+5)}\right)$$

$$-\frac{6^{l}}{5\ldots.(4l+1)}\cdot\sin\frac{m\pi}{(4l+5)}=0.$$

Therefore, the above equation is approximately equivalent to

$$\frac{1}{5.9\ldots.(4l+5)}\cdot\pi\,(4l+5+m)\left[\frac{6^{l}-6}{5}\right]$$

$$-\frac{6^{l}\cdot m\pi}{5\ldots.(4l+5)}=0$$

whence we have

$m=l$ as a first approximation.

§ 3

Zeros of Lerch's function:

$$L(x)\equiv\sum_{0}^{\infty}\frac{r^{n}\cos(a_{n}\pi x)}{a_{n}}$$

11. Let $p_0, p_1, p_2, \ldots p_n, \ldots$ be a sequence of odd integers, not tending to a finite limit with n tending to infinity and let $r>1$, then Lerch's function is given by

$$L(x)=\sum_{n=0}^{\infty}\frac{r^{n}\cos(a_{n}\pi x)}{a_{n}};$$

where a_n denotes the product $p_0\, p_1\, p_2 \ldots p_n$.

It is clear that all odd integers can be classified under two heads :—

(1) those given by $4m+3$,

(2) those given by $4m+1$,

where m may be any positive integer or zero.

We will therefore investigate zeros of the three types of functions as derived from the two classes of odd integers, *i.e.*,

A where $p_0, p_1, p_2 \ldots$ are all of the form $4m+3$,

B where $p_0, p_1, p_2 \ldots$ are all of the form $4m+1$,

C where some p's are of the form $4m+3$ and the others of the form $4m+1$.

12. *Lemma.*

Let c_r denote $\dfrac{\sin \theta_r}{\theta_r}$, where $0 < \theta_r < \dfrac{\pi}{2}$. Then it is obvious that c_r lies between 1 and $\dfrac{2}{\pi}$; and it is also clear that as θ_r increases, c_r diminishes. Further, for $\theta_r \equiv \dfrac{3\pi/2}{4r+3}$

$$c_r > 1 - \frac{\theta_r^{\,2}}{\lfloor 3} > 1 - \frac{4}{\lfloor 3\, r^2},$$

because

$$\theta_r < \frac{2}{r}, \text{ as } \theta_r \equiv \frac{3\pi/2}{4r+3}.$$

CLASS A.

13. *There is a zero of $L(x)$ between*

$$\frac{1}{2} + \frac{3/2}{p_0\, p_1}, \text{ and } \frac{1}{2} + \frac{3/2}{p_0\, p_1\, p_2}.$$

Proof :—

$$L\left(\frac{1}{2} + \frac{3/2}{p_0\, p_1}\right) = \frac{\sin \dfrac{3\pi/2}{p_1}}{p_0} - \frac{r}{p_0 p_1} \sin (3\pi/2) + \frac{r^2}{p_0 p_1 p_2} \sin\left(\frac{\pi}{2}\right) - \ldots$$

$$= \frac{\sin \dfrac{3\pi/2}{p_1}}{p_0} + \frac{r}{p_0 p_1} + \frac{r^2}{p_0 p_1 p_2} + \ldots\ldots$$

$$> 0.$$

Also

$$L\left(\frac{1}{2}+\frac{3/2}{p_0p_1p_2}\right)=\frac{\sin\dfrac{3\pi/2}{p_1p_2}}{p_0}-\frac{r}{p_0p_1}\sin\frac{3\pi/2}{p_2}+\frac{r^2}{p_0p_1p_2}\sin(3\pi/2)$$

$$-\frac{r^3}{p_0p_1p_2p_3}\sin\left(\frac{\pi}{2}\right)+\ldots\ldots$$

$$=\frac{\sin\dfrac{3\pi/2}{p_1p_2}}{p_0}-\frac{r}{p_0p_1}\sin\frac{3\pi/2}{p_2}-\frac{r^2}{p_0p_1p_2}-\frac{r^3}{p_0p_1p_2p_3}-\ldots(1)$$

Let k_l stand for $\frac{\sin\theta_l}{\theta_l}$, where θ_l is the angle which appears with r^l in the expression (1).

Therefore the expression (1) $=\frac{3\pi/2}{p_0p_1p_2}k_0-r\,k_1\frac{3\pi/2}{p_0p_1p_2}$

$$-\frac{r^2}{p_0p_1p_2}-\frac{r^3}{p_0p_1p_2p_3}\ldots\ldots$$

$$=\frac{(3\pi/2)\{k_0-r\,k_1\}-r^2}{p_0p_1p_2}-\frac{r^3}{p_0p_1p_2p_3}\ldots\ldots \qquad (2)$$

Now put $r=1+a$, where a is a small positive number.

Then the expression (2) $<\frac{3\pi/2}{p_0p_1p_2}\left\{k_0-k_1-k_1\,a\right\}-\frac{1+2a}{p_0p_1p_2}-\ldots$

(neglecting a^2, etc.,...)

$$=\frac{(3\pi/2)\{k_0-k_1\}-1-(k_1+2)a}{p_0p_1p_2}-\ldots\ldots$$

$$<\frac{(3\pi/2)\{1-k_1\}-1-(k_1+2)a}{p_0p_1p_2}$$

$$<0,\text{ for }(3\pi/2)\{1-k_1\}<1\quad\text{(by Lemma)}.$$

Thus it is proved that there is a root of $L(x)=0$ between

$$\frac{1}{2}+\frac{3/2}{p_0p_1}\text{ and }\frac{1}{2}+\frac{3/2}{p_0p_1p_2}.$$

14. *There is a zero of $L(x)$ between*

$$\frac{1}{2}+\frac{3/2}{p_0p_1p_2p_3} \text{ and } \frac{1}{2}+\frac{3/2}{p_0p_1p_2p_3p_4}.$$

Proof :—

$$L\left(\frac{1}{2}+\frac{3/2}{p_0p_1p_2p_3}\right)=\frac{\sin\dfrac{3\pi/2}{p_1p_2p_3}}{p_0}-\frac{r}{p_0p_1}\sin\frac{3\pi/2}{p_2\,p_3}+\frac{r^2}{p_0p_1p_2}\sin\frac{3\pi/2}{p_3}$$

$$-\frac{r^3}{p_0p_1p_2p_3}\sin(3\pi/2)+\frac{r^4}{p_0p_1p_2p_3p_4}\sin(\pi/2)-\ldots$$

$$=\frac{\sin\dfrac{3\pi/2}{p_1p_2p_3}}{p_0}-\frac{r}{p_0p_1}\sin\frac{3\pi/2}{p_2\,p_3}+\frac{r^2}{p_0p_1p_2}\sin\frac{3\pi/2}{p_3}$$

$$+\frac{r^3}{p_0p_1p_2p_3}+\frac{r^4}{p_0p_1p_2p_3p_4}+\ldots\ldots$$

$$>\frac{3}{p_0p_1p_2p_3}-\frac{r\,(3\pi/2)}{p_0p_1p_2p_3}+\frac{3r^2}{p_0p_1p_2p_3}+\frac{r^3}{p_0p_1p_2p_3}+\ldots\ldots$$

$$>\frac{3(1+r^2)-(3\pi/2)r+r^3}{p_0p_1p_2p_3}+\ldots\ldots \qquad (1)$$

Now put $r=1+\alpha$, where $\alpha>0$, and small;

$$\text{Then the expression (1)}=\frac{3\,(2+\alpha)-(3\pi/2)\,(1+\alpha)+(1+3\alpha)}{p_0p_1p_2p_3}+\ldots$$

$$=\frac{7-(3\pi/2)+\alpha\,(6-3\pi/2)}{p_0p_1p_2p_3}+\ldots$$

$$>0.$$

Also

$$L\left(\frac{1}{2}+\frac{3/2}{p_0p_1p_2p_3p_4}\right)=\frac{\sin\dfrac{3\pi/2}{p_1p_2p_3p_4}}{p_0}-\frac{r}{p_0p_1}\sin\frac{3\pi/2}{p_2p_3p_4}$$

$$+\frac{r^2}{p_0p_1p_2}\sin\frac{3\pi/2}{p_3p_4}-\frac{r^3}{p_0p_1p_2p_3}\sin\frac{3\pi/2}{p_4}+\frac{r^4}{p_0p_1p_2p_3p_4}\sin(3\pi/2)$$

$$-\frac{r^5}{p_0\cdots p_5}\sin(\pi/2)+\ldots$$

$$=\frac{\sin\dfrac{3\pi/2}{p_1p_2p_3p_4}}{p_0}-\frac{r}{p_0p_1}\sin\frac{3\pi/2}{p_2p_3p_4}+\frac{r^2}{p_0p_1p_2}\sin\frac{3\pi/2}{p_3p_4}$$

$$-\frac{r^3}{p_0p_1p_2p_3}\sin\frac{3\pi/2}{p_4}-\frac{r^4}{p_0p_1p_2p_3p_4}-\ldots \tag{2}$$

$$=\frac{(3\pi/2)\{k_0-k_1r+k_2r^2-k_3r^3\}-r^4}{p_0p_1p_2p_3p_4}-\ldots$$

(θ_l denoting the angle appearing with r^l above)

$$=\frac{(3\pi/2)\{k_0+k_2r^2-r(k_1+k_3r^2)\}-r^4}{p_0p_1p_2p_3p_4}-\ldots$$

$$<\frac{(3\pi/2)\{(1+r^2)(1-rk_3)\}-r^4}{p_0p_1p_2p_3p_4}-\ldots \tag{3}$$

Now, in the above put $1+\alpha$ for r, where α is small and > 0.

Therefore the expression (2) becomes

$$<\frac{(3\pi/2)\{2(1+\alpha)(1-k_3-k_3\alpha)\}-1-4\alpha}{p_0p_1p_2p_3p_4} \quad \text{(neglecting } \alpha^2\text{, etc.,...)}$$

$$<\frac{3\pi\{(1+\alpha)(1-k_3-k_3\alpha)\}-1-4\alpha}{p_0p_1p_2p_3p_4}-\ldots$$

$$<\frac{3\pi\{1-k_3\}-1-3\pi\{2k_3-1\}\cdot\alpha-4\alpha}{p_0p_1p_2p_3p_4}. \tag{4}$$

But $k_3 > 1-\dfrac{2}{3.4^2}$ (by Lemma).

Therefore, $3\pi(1-k_3) < 1$.

Further, $2k_3 > 1$.

Therefore the expression (4) < 0.

Thus it is proved that there is a root of

$L(x)=0$, between $\frac{1}{2}+\frac{3/2}{p_0p_1p_2p_3}$ and $\frac{1}{2}+\frac{3/2}{p_0p_1p_2p_3p_4}$.

15. *Generally there is a root of $L(x)=0$ between*

$$\frac{1}{2}+\frac{3/2}{p_0\cdots p_{2t+1}} \text{ and } \frac{1}{2}+\frac{3/2}{p_0\cdots p_{2t+2}}.$$

Proof :—

$$L\left(\frac{1}{2}+\frac{3/2}{p_0\cdots p_{2t+1}}\right)$$

$$=\frac{\sin\dfrac{3\pi/2}{p_1\cdots p_{2t+1}}}{p_0}-\frac{r}{p_0p_1}\sin\frac{3\pi/2}{p_2\cdots p_{2t+1}}+\frac{r^2}{p_0p_1p_2}\sin\frac{3\pi/2}{p_3\cdots p_{2t+1}}-\dots$$

$$\dots+\frac{r^{2t}}{p_0\cdots p_{2t}}\sin\frac{3\pi/2}{p_{2t+1}}-\frac{r^{2t+1}}{p_0\cdots p_{2t+1}}\sin(3\pi/2)$$

$$+\frac{r^{2t+2}}{p_0\cdots p_{2t+2}}\sin\left(\frac{\pi}{2}\right)+\cdots$$

$$=\frac{\sin\dfrac{3\pi/2}{p_1\cdots p_{2t+1}}}{p_0}-\frac{r}{p_0p_1}\sin\frac{3\pi/2}{p_2\cdots p_{2t+1}}+\cdots$$

$$+\frac{r^{2t}}{p_0\cdots p_{2t}}\sin\frac{3\pi/2}{p_{2t+1}}+\frac{r^{2t+1}}{p_0\cdots p_{2t+1}}+\frac{r^{2t+2}}{p_0\cdots p_{2t+2}}+\dots$$

$$=\frac{(3\pi/2)\{k_0-k_1r+k_2r^2-\dots+k_{2t}.r^{2t}\}+r^{2t+1}}{p_0\cdots p_{2t+1}}+\cdots$$

(θ_l denoting the angle appearing with r^l above)

$$>\frac{(3\pi/2)\{(k_{2t-2}-k_1r)(1+r^2+\dots+r^{2t-2})+k_{2t}r^{2t}\}}{p_0\cdots p_{2t+1}}+\frac{r^{2t+1}}{p_0\cdots p_{2t+1}}+\dots$$

$$>\frac{\{r^{2t+1}+(3\pi/2)k_{2t}r^{2t}\}-(3\pi/2)(k_1r-k_{2t-2})\times(1+r^2+\dots+r^{2t-2})}{p_0\cdots p_{2t+1}}\dots(1)$$

Now, put $1+\alpha$ for r, $\alpha > 0$ and small.

Then the expression (1)

$$> \frac{\{1+(2t+1)\alpha+(3\pi/2)k_{2t}(1+2t\cdot\alpha)\}}{p_0\cdots p_{2t+1}}$$

$$-\frac{(k_1+k_1\alpha-k_{2t-2})\{t+t(t-1)\alpha\}(3\pi/2)}{p_0\cdots p_{2t+1}}+\ldots$$

$$> \frac{\{1+(3\pi/2)k_{2t}+\alpha(2t+1+3\pi t)\}}{p_0\cdots p_{2t+1}}$$

$$-\frac{(3\pi/2)[(k_1-k_{2t-2})t+(k_1-k_{2t-2}).t.(t-1)\alpha+k_1 t\alpha]}{p_0\cdots p_{2t+1}}+\ldots > 0$$

if $\quad 1+(3\pi/2)k_{2t}-(3\pi/2)t(k_1-k_{2t-2}) > 0$

i.e., if $\quad 4-(3\pi/2)t(1-k_{2t-2}) > 0$

i.e., if $\quad \dfrac{8}{3\pi.t} > 1-k_{2t-2}$.

But this is so, as can be easily verified remembering that

$$k_{2t-2} > 1-\frac{1}{6(t-1)^2} > 1-\frac{8}{3\pi t}, \text{ for } t\geq 2.$$

Also

$$L\left(\frac{1}{2}+\frac{3/2}{p_0\cdots p_{2t+2}}\right)=\frac{\sin\dfrac{3\pi/2}{p_1\cdots p_{2t+2}}}{p_0}-\frac{r}{p_0p_1}\sin\frac{3\pi/2}{p_2\cdots p_{2t+2}}+\ldots$$

$$+\frac{r^{2t}}{p_0\cdots p_{2t}}\sin\frac{3\pi/2}{p_{2t+1}p_{2t+2}}-\frac{r^{2t+1}}{p_0\cdots p_{2t+1}}\sin\frac{3\pi/2}{p_{2t+2}}$$

$$+\frac{r^{2t+2}}{p_0\cdots p_{2t+2}}\sin(3\pi/2)-\ldots$$

$$=\frac{(3\pi/2)\{k_0-k_1r+k_2r^2-\ldots+r^{2t}.k_{2t}-r^{2t+1}.k_{2t+1}\}-r^{2t+2}}{p_0\cdots p_{2t+2}}\ldots$$

(θ_l denoting the angle appearing with r^l above)

$$=\frac{(3\pi/2)\{k_0+k_2r^2+\ldots+k_{2t}r^{2t}-r(k_1+k_3r^2+\ldots+r^{2t}.k_{2t+1}.)\}-r^{2t+2}}{p_0\ldots p_{2t+2}}$$

$$<\frac{(3\pi/2)\{(1+r^2+\ldots+r^{2t})(k_0-k_{2t+1}r)\}-r^{2t+2}}{p_0\ldots p_{2t+2}}. \qquad (1)$$

Now put $r=1+\alpha$, where $\alpha>0$ and small.

Thus the expression (1)

$$<\frac{(3\pi/2)\{t+1+t(t+1).\alpha\}(1-k_{2t+1}-k_{2t+1}.\alpha)}{p_0\ldots p_{2t+2}}$$

$$-\frac{1+(2t+2)\alpha}{p_0\ldots p_{2t+2}}, \text{ (neglecting, } \alpha^2,\ \alpha^3, \text{ etc.)}$$

$$<\frac{P}{Q},$$

where $(3\pi/2)\{(t+1)(1-k_{2t+1})-(t+1)(k_{2t+1})\alpha+t(t+1)(1-k_{2t+1})\alpha\}$
$-1-(2t+2)\alpha$

and $p_0\ldots p_{2t+2}$ are equal to P and Q respectively.

Therefore the expression (1) <0

if $(3\pi/2)\ \{(t+1)\ (1-k_{2t+1})\}<1$

i.e., if, $1-k_{2t+1}<\dfrac{2}{3\pi(t+1)}$

i.e., if $k_{2t+1}>1-\dfrac{2}{3\pi(t+1)}$.

But $\quad k_{2t+1}>1-\dfrac{2}{3(2t+1)^2}>1-\dfrac{2}{3\pi(t+1)}$ for $t>0$.

Hence * it is proved that there is a root of $L(x)=0$, between

$$\frac{1}{2}+\frac{3/2}{p_0\ldots p_{2t+1}} \text{ and } \frac{1}{2}+\frac{3/2}{p_0\ldots p_{2t+2}};$$

where t may have any positive integral value including zero.

* The case $t=0$ has been considered in Art. 13.

We have treated r, in the preceding discussion, to be just greater than 1; it is needless to add that it is much simpler to find zeros when r is sufficiently large, as has been shewn in the very beginning of this paper while discussing zeros of Darboux's functions, where r is taken to be 6.

Class B.

16. In this section, we will briefly consider the second type* of odd integers, *i. e.*, where the p's are all of the form $4m+1$, where m may be any positive integer including zero.

There is a zero of $L_B(x)$ between

$$\frac{1}{2}+\frac{1}{p_0p_1\cdots p_k} \text{ and } \frac{1}{2}+\frac{3/2}{p_0p_1\cdots p_k},$$

$$\textit{provided } r \geq 1+\frac{3\pi}{2}.$$

Proof :—

$$-L_B\left(\frac{1}{2}+\frac{1}{p_0p_1\cdots p_k}\right)=\frac{\sin\dfrac{\pi}{p_1\cdots p_k}}{p_0}+\frac{r\sin\dfrac{\pi}{p_2\cdots p_k}}{p_0p_1}+\dots$$

$$+\frac{r^{k-1}}{p_0\cdots p_{k-1}}\sin\frac{\pi}{p_k}+0+\dots$$

$$>0.$$

Also

$$-L_B\left(\frac{1}{2}+\frac{3/2}{p_0\cdots p_k}\right)=\frac{\sin\dfrac{3\pi/2}{p_1\cdots p_k}}{p_0}+\frac{r}{p_0p_1}\sin\frac{3\pi/2}{p_2\cdots p_k}+\dots$$

$$+\frac{r^{k-1}}{p_0\cdots p_{k-1}}\sin\frac{3\pi/2}{p_k}+\frac{r^k}{p_0\cdots p_k}\sin(3\pi/2)+\dots$$

$$=\frac{\sin\dfrac{3\pi/2}{p_1\cdots p_k}}{p_0}+\dots\dots+\frac{r^{k-1}}{p_0\cdots p_{k-1}}\sin\frac{3\pi/2}{p_k}-\frac{r^k}{p_0\cdots p_k}-\dots$$

* The function is here denoted by $L_B(x)$.

$$< \frac{(3\pi/2)(1+r+r^2+\ldots\ldots+r^{k-1})-r^k}{p_0\ldots p_k}$$

$$= \frac{(3\pi/2)\left\{\dfrac{r^k-1}{r-1}\right\}-r^k}{p_0\ldots p_k} < 0\,;$$

since $r \geq 1+\dfrac{3\pi}{2}$.

Thus it is proved that there is a root of $L_B\,(x)=0$, between

$$\frac{1}{2}+\frac{1}{p_0\ldots p_k} \text{ and } \frac{1}{2}+\frac{3/2}{p_0\ldots p_k}\,;$$

for every positive integral value of k, provided $r \geq 1+\dfrac{3\pi}{2}$.

Class C.

17. Here we will consider only one or two types of variation. First, we consider the case * where p_0, p_2, $p_4\ldots p_{2n}$ are all of the form $4m+3$ and p_1, $p_3,\ldots p_{2n+1}$ are all of the form $4m+1$, where m may have any positive integral value.

There is a zero of $L_{C_1}\,(x)$ between

$$\frac{1}{2}+\frac{3/2}{p_0\cdots p_{4n}} \text{ and } \frac{1}{2}+\frac{3/2}{p_0\ldots p_{4n+2}}$$

where n may have any positive integral value.

Proof :—

$$L_{C_1}\left(\frac{1}{2}+\frac{3/2}{p_0\cdot\cdot p_{4n}}\right)=\frac{\sin\dfrac{3\pi/2}{p_1\ldots p_{4n}}}{p_0}+\frac{r}{p_0p_1}\sin\frac{3\pi/2}{p_2\ldots p_{4n}}$$

$$-\frac{r^2}{p_0p_1p_2}\sin\frac{3\pi/2}{p_3\ldots p_{4n}}-\frac{r^3}{p_0p_1p_2p_3}\sin\frac{3\pi/2}{p_4\ldots p_{4n}}$$

* The function in the present case is denoted by $L_{C_1}\,(x)$.

$$+\dots-\frac{r^{4n-2}}{p_0\cdots p_{4n-2}}\sin\frac{3\pi/2}{p_{4n-1}\cdot p_{4n}}-\frac{r^{4n-1}}{p_0\dots p_{4n-1}}\sin\frac{3\pi/2}{p_{4n}}$$

$$+\frac{r^{4n}}{p_0\dots p_{4n}}\sin(3\pi/2)+\dots$$

$$=\frac{\sin\dfrac{3\pi/2}{p_1\dots p_{4n}}}{p_0}+\frac{r}{p_0p_1}\sin\frac{3\pi/2}{p_2\dots p_{4n}}-\frac{r^2}{p_0p_1p_2}\sin\frac{3\pi/2}{p_3\dots p_{4n}}$$

$$-\frac{r^3}{p_0p_1p_2p_3}\sin\frac{3\pi/2}{p_4\dots p_{4n}}+\dots-\frac{r^{4n-2}}{p_0\dots p_{4n-2}}\sin\frac{3\pi/2}{p_{4n-1}\cdot p_{4n}}$$

$$-\frac{r^{4n-1}}{p_0\dots p_{4n-1}}\sin\frac{3\pi/2}{p_{4n}}-\frac{r^{4n}}{p_0\dots p_{4n}}-\dots$$

$$=\frac{(3\pi/2)\{k_0+rk_1-r^2k_2-\dots-r^{4n-2}k_{4n-2}-r^{4n-1}k_{4n-1}\}}{p_0\dots p_{4n}}$$

$$-\frac{r^{4n}}{p_0\dots p_{4n}}-\dots$$

$$<\frac{3\pi/2}{p_0\dots p_{4n}}\{(1+r)(1+r^4+\dots+r^{4n-4})\}(1-k_{4n-1}r^2)-\frac{r^{4n}}{p_0\dots p_{4n}}-\dots$$

$$<0$$

Since, by putting $r=1+a$, where a is a small positive quantity, we find that

$$3\pi n(1-k_{4n-1})-1<0.$$

Also

$$L_{U_1}\left(\frac{1}{2}+\frac{3/2}{p_0\dots p_{4n+2}}\right)$$

$$=\frac{\sin\dfrac{3\pi/2}{p_1\dots p_{4n+2}}}{p_0}+\frac{r}{p_0p_1}\sin\frac{3\pi/2}{p_2\dots p_{4n+2}}-\frac{r^2}{p_0p_1p_2}\sin\frac{3\pi/2}{p_0\dots p_{4n+2}}$$

$$-\frac{r^3}{p_0p_1p_2p_3}\sin\frac{3\pi/2}{p_4\cdots p_{4n+2}}+\ldots-\frac{r^{4n-2}}{p_0\cdots p_{4n-2}}\sin\frac{3\pi/2}{p_{4n-1}\cdots p_{4n+2}}$$

$$-\frac{r^{4n-1}}{p_0\cdots p_{4n-1}}\sin\frac{3\pi/2}{p_{4n}\cdots p_{4n+2}}+\frac{r^{4n}}{p_0\cdots p_{4n}}\sin\frac{3\pi/2}{p_{4n+1}\cdot p_{4n+2}}$$

$$+\frac{r^{4n+1}}{p_0\cdots p_{4n+1}}\sin\frac{3\pi/2}{p_{4n+2}}-\frac{r^{4n+2}}{p_0\cdots p_{4n+2}}\sin(3\pi/2)-\ldots$$

$$=\frac{\sin\dfrac{3\pi/2}{p_1\cdots p_{4n+2}}}{p_0}+\frac{r}{p_0p_1}\sin\frac{3\pi/2}{p_2\cdots p_{4n+2}}-\frac{r^2}{p_0p_1p_2}\sin\frac{3\pi/2}{p_3\cdots p_{4n+2}}$$

$$-\frac{r^3}{p_0p_1p_2p_3}\sin\frac{3\pi/2}{p_4\cdots p_{4n+2}}+\ldots+\frac{r^{4n}}{p_0\cdots p_{4n}}\sin\frac{3\pi/2}{p_{4n+1}p_{4n+2}}$$

$$+\frac{r^{4n+1}}{p_0\cdots p_{4n+1}}\sin\frac{3\pi/2}{p_{4n+2}}+\frac{r^{4n+2}}{p_0\cdots p_{4n+2}}+\ldots$$

$$>\frac{(3\pi/2)\{(1+r)(1+r^4+\ldots+r^{4n})k_{4n+1}-(1+r)r^2(1+r^4+\ldots+r^{4n-4})\}}{p_0\cdots p_{4n+2}}$$

$$+\frac{r^{4n+2}}{p_0\cdots p_{4n+2}}+\ldots$$

$$>\frac{3\pi/2}{p_0\cdots p_{4n+2}}(1+r)\{-(1+r^4+\ldots+r^{4n-4})(r^2-k_{4n+1})+r^{4n}k_{4n+1}\}$$

$$+\frac{r^{4n+2}}{p_0\cdots p_{4n+2}}+\ldots$$

$$>0.$$

Since, by putting $r=1+a$, where $a>0$ and small, we find after a little calculation that

$$-3\pi n(1-k_{4n+1})+3\pi k_{4n+1}+1>0.$$

Hence it is proved that there is a root of $L_{C_1}(x)=0$, between

$$\frac{1}{2}+\frac{3/2}{p_0\cdots p_{4n}},\ \text{and}\ \frac{1}{2}+\frac{3/2}{p_0\cdots p_{4n+2}};$$

for every positive integral value of n.

18. A second variation is the following case * where p_0 is of the form $4m+1$

p_1 and p_2......$4m+3$,

p_3...... $4m+1$,

p_4 and p_5......$4m+3$,

and so on.

There is a zero of $L_{C_2}(x)$ between

$$\frac{1}{2}+\frac{3/2}{p_0\dots p_{3n+1}}, \text{ and } \frac{1}{2}+\frac{3/2}{p_0\dots p_{3n+2}};$$

provided $r \geq 1{\cdot}5$.

Proof :—

$$L_{C_2}\left(\frac{1}{2}+\frac{3/2}{p_0\dots p_{3n+1}}\right)=\frac{-\sin\dfrac{3\pi/2}{p_0\dots p_{3n+1}}}{p_0}+\frac{r}{p_0p_1}\sin\frac{3\pi/2}{p_2\dots p_{3n+1}}$$

$$-\dots-\frac{r^{3n}}{p_0\dots p_{3n}}\sin\frac{3\pi/2}{p_{3n+1}}-\frac{r^{3n+1}}{p_0\dots p_{3n+1}}-\dots$$

$$=\frac{3\pi/2}{p_0\dots p_{3n+1}}\left\{-k_0+k_1r-k_2r^2-k_3r^3+k_4r^4-\dots\right\}-\frac{r^{3n+1}}{p_0\dots p_{3n+1}}-\dots$$

$$<\frac{3\pi/2}{p_0\dots p_{3n+1}}\left\{r(1+r^3+\dots+r^{3n-3})(1-k_{3n-1}r-k_{3n}r)-k_0\right\}-\dots$$

< 0, for all values of $r > 1$.

Also

$$L_{C_2}\left(\frac{1}{2}+\frac{3/2}{p_0\dots p_{3n+2}}\right)$$

$$=\frac{3\pi/2}{p_0\dots p_{3n+2}}\left\{-k_0+k_1r-k_2r^2-k_3r^3+k_4r^4-\dots+k_{3n+1}r^{3n+1}\right\}$$

* The function is denoted by $L_{C_2}(x)$.

$$+\frac{r^{3n+2}}{p_0\cdots p_{3n+2}}+\ldots$$

$$>\frac{3\pi/2}{p_0\cdots p_{3n+2}}\left\{-1+r(1+r^3+\ldots r^{3n-3})(k_{3n+1}-r-r^2)+k_{3n+1}r^{3n+1}\right\}$$

$$>3\pi[-(r^3-1)+r(r^{3n}-1)(k_{3n+1}-r-r^2)+k_{3n+1}r^{3n+1}(r^3-1)]$$
$$+2r^{3n+2}(r^3-1)$$

$$>0,$$

if, $3\pi k_{3n+1}r^2+2r^3>(3\pi+2)+3\pi r$, *i.e.*, if $r\geq 1{\cdot}5$.

Thus it is proved that there is a root of $L_{C_2}(x)=0$, between

$$\frac{1}{2}+\frac{3/2}{p_0p_1\cdots p_{3n+1}},\text{ and }\frac{1}{2}+\frac{3/2}{p_0p_1\cdots p_{3n+2}};$$

provided $r\geq 1{\cdot}5$.

It is to be noted, however, that r may have a lesser value, like 1·45, satisfying the above condition, provided we commence the sequence of p's from a bigger number, *i.e.*, when $3\pi k_{3n+1}$ approximates to 9·4 at least.

Conclusion.

19. Hitherto, our attention has been mainly directed to search out one single limiting point of zeros of non-differentiable functions. We will conclude this paper, with an investigation of some other limiting points, which are, however, finite in number, in the interval (0, 1).

It is clear that the equation,

$$\overline{D}(x)=0,$$

remains unaltered by one or the other of the following substitutions

$$\tfrac{1}{5}+x,\quad \tfrac{2}{5}+x,\quad \tfrac{3}{5}+x,\quad \tfrac{4}{5}+x.$$

Therefore $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, and $\frac{4}{5}$ are naturally limiting points of zeros of $\overline{D}(x)$.

20. We know that the non-differentiable character of functions, representable by infinite series, is unaffected, by the omission of a finite number of terms, from the respective series; hence, let us consider the following non-differentiable function,

$$\overline{D}_1(x) = \frac{6^2 \sin 5.9.\pi x}{5.9} + \ldots$$

which is obtained by omitting the first term of the series for $D(x)$. Here, again, we find that the equation,

$$\overline{D}_1(x) = 0,$$

remains unchanged by the substitutions,

$$\frac{1}{5} + x, \ldots \frac{1}{9} + x, \ldots \frac{1}{5.9} + x, \ldots \frac{44}{5.9} + x.$$

Therefore,

$$\frac{1}{5}, \ldots \frac{1}{9}, \ldots \frac{1}{5.9}, \ldots \frac{44}{5.9},$$

are all limiting points of zeros of $\overline{D}_1(x)$.

Precisely, in the same manner, we can find out any number of limiting points, though finite, in the case of other non-differentiable functions, such as Darboux's cosine type, Lerch's function, Weierstrass's function, etc., by leaving out a certain number of terms, from their respective series.

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2

On the Unenumerable Zeros of Singh's Non-differentiable Function

By

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Introduction.

1. In a paper recently published in the *Proceedings of the Benares Mathematical Society, Vol XI,* Prof. G. Prasad has located zeros of Weierstrass's non-differentiable function.* The zeros that have been found out by him form an enumerable set with the point $x=\frac{1}{2}$, as a limiting point. The present investigation was suggested by Prof. Prasad's work.

The object of this paper is to study the zeros of a class of Non-differentiable functions defined by me in a paper published in the *Annals of Mathematics,* Vol. 28, (1927), pp. 472-76. As these functions are arithmetically defined, the location of the zeros presents little difficulty. It has been shown that the set of the zeros of any of these functions $\theta(t)$ possesses the following properties:

(*a*) *it is unenumerable;* †
(*b*) *it is perfect ;*
(*c*) *it has zero measure.*

It has been further pointed out that the roots of the equation

$$\theta(t)=c, \quad 0<c<1,$$

form a set similar to the set of the zeros.

* $Y = \sum a^n \cos(b^n x\pi)$, subject to the condition $a<1$, $ab>1+\frac{3\pi}{2}$, (b an odd integer).

† The function $F(t)=\int_0^t \theta(t)\,dt$ does not possess proper maxima or minima at the points where $F'(t)=0$, for the proper maxima or minima form an enumerable set. *Cf.* Hobson, *Theory of Functions*, Vol. I, (3rd ed.), p. 350.

Definition of the Class of Functions.

2. Let $b_1, b_2, b_3, \ldots b_m$ denote the m odd numbers 3, 5, 7,...$(2m+1)$ respectively. Then any number in the interval (0, 1) can be represented as

$$t = \frac{a_1}{3} + \frac{a_2}{3.5} + \frac{a_3}{3.5.7} + \ldots + \frac{a_m}{3.5.\ldots(2m+1)}$$

$$+ \frac{a_{m+1}}{3^2.5.7.\ldots.(2m+1)} + \ldots + \frac{a_{2m}}{3^2.5^2.\ldots.(2m+1)^2} + \ldots$$

where the a's are positive integers such that

$$0 \leq a_n \leq b_n - 1, \qquad (n=1, 2, 3, \ldots).$$

Let the quantities $x_1, x_2, \ldots x_m$ be defined as follows:

$$x_1 = \theta_1(t) = \frac{c_{1,1}}{3} + \frac{c_{1,2}}{3^2} + \frac{c_{1,3}}{3^3} + \ldots\ldots;$$

$$x_2 = \theta_2(t) = \frac{c_{2,1}}{5} + \frac{c_{2,2}}{5^2} + \frac{c_{2,3}}{5^3} + \ldots\ldots;$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots;$$

$$x_m = \theta_m(t) = \frac{c_{m,1}}{(2m+1)} + \frac{c_{m,2}}{(2m+1)^2} + \frac{c_{m,3}}{(2m+1)^3} + \ldots$$

Here the c's are defined as below:

$$c_{1,1} = a_1, \quad c_{1,2} = P_1^{a_2 a_3 + \ldots + a_m}(a_{m+1}),$$

$$c_{1,3} = P_1^{a_2 + \ldots + a_m + a_{m+2} + \ldots a_{2m}}(a_{2m+1}), \ldots$$

$$c_{2,1} = P_2^{a_1}(a_2), \quad c_{2,2} = P_2^{a_1 + a_3 + a_4 + \ldots + a_{m+1}}(a_{m+2}), \ldots$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$c_{m,1} = P_m^{a_1 + a_2 + \ldots + a_{m-1}}(a_m), \quad c_{m,2} = P_m^{a_1 + \ldots + a_{m-1} + a_{m+1} + \ldots + a_{2m-1}}(a_{2m}), \ldots$$

where $P_r^k(a)$ denotes a or $(2r-a)$ according as k is even or odd $(r=1, 2,\ldots m)$.

It can be easily proved that the set of the zeros of any of these functions, possesses the properties stated in the introduction. The method of proof is illustrated by the following example, in which we take $m=2$, and $r=1$.

THE PARTICULAR CASE, $m=2$, $r=1$.

3. Let any point t in the interval (0, 1) be represented as

$$t = \frac{a_1}{3} + \frac{a_2}{3.5} + \frac{a_3}{3^2.5} + \frac{a_4}{3^2.5^2} + \ldots$$

where the a's are positive integers such that $0 \leq a_{2n} \leq 4$ and $0 \leq a_{2n+1} \leq 2$, $(n=0, 1,2,\ldots)$.

Corresponding to t, let a number x be defined as

$$x \equiv \theta(t) = \frac{c_1}{3} + \frac{c_2}{3^2} + \frac{c_3}{3^3} + \ldots\ldots$$

where $c_1=a_1$, $c_2=P_1^{a_2}(a_3),\ldots$

$$c_{n+1}=P_1^{a_2+a_4+\ldots+a_{2n}}(a_{2n+1}),\ldots\ldots$$

THE SET OF THE ZEROS.

The set of the zeros S of $\theta(t)$ is defined by those numbers t, in (0, 1 which are such that

$$a_1=0, \text{ and } a_{2n+1}=0 \text{ or } 2$$

according as

$$a_2+a_4+\ldots+a_{2n}$$

is even or odd, for when this is the case

$$c_1=0 \text{ and } c_{n+1}=P_1^{a_2+a_4+\ldots a_{2n}}(a_{2n+1})=0$$

for all values of n, and, therefore, for all such values of t

$$\theta(t)=0.$$

It is easy to see that

$$c_{n+1}=P_1^{a_2+a_4+\ldots+a_{2n}}(a_{2n+1})$$

will be zero if, and only if, t satisfies the conditions given above, otherwise c_n will be either 1 or 2. Consequently it follows that *S is the set of all the zeros of $\theta(t)$*.

The Properties of the set of the zeros.

4. We now proceed to prove that the set of the zeros S possesses the properties mentioned in the introduction.

(a) *S is unenumerable.* To prove this it will be sufficient to show that the part set S_1, defined as below, is unenumerable.

Let the points t of the set S_1, have the following representation :

$$t=\frac{0}{3}+\frac{a_2}{3.5}+\frac{0}{3^2.5}+\frac{a_4}{3^2.5^2}+\frac{0}{3^3.5^2}+\ldots$$

$$+\frac{0}{3^n.5^{n-1}}+\frac{a_{2n}}{3^n.5^n}+\ldots\ldots$$

where $a_{2n}=0$, 2 or 4, $(n=1, 2,\ldots)$.

It is easy to see that the set S_1 is contained in the set S. Further, as the different members of the set S_1 are obtained by giving to the a_{2n}'s, one of the three values, 0, 2 or 4, the numbers t of S_1 can be placed into one-to-one correspondence with the numbers of the continuum expressed in the scale of 3. *Thus S is unenumerable.*

(b) *S is perfect.* To prove this we first show that S is dense-in-itself, and then that it is closed.

Let

$$t=\frac{0}{3}+\frac{a_2}{3.5}+\frac{a_3}{3^2.5}+\frac{a_4}{3^2.5^2}++\ldots\frac{a_{2n}}{3^n.5^n}+\cdots$$

be an arbitrarily chosen point of S. It will now be shown that t is a limiting point of S.

The point t must belong to at least one of the following two classes, such that in the representation of t either

(i) an infinite number of a_{2n}'s are ≥ 2,

or (ii) an infinite number of a_{2n}'s are < 2

If t belongs to class (i), the point

$$t_n = \frac{0}{3} + \frac{a_2}{3.5} + \frac{a_3}{3^2.5} + \cdots + \frac{a'_{2n}}{3^n.5^n} + \frac{a_{2n+1}}{3^{n+1}.5^n} + \cdots$$

which differs from t at the $2n$th place only, will be a point of the set S, if

$$a'_{2n} = a_{2n} - 2.$$

But

$$\mid t_n - t \mid = \frac{2}{3^n.5^n},$$

tends to zero when n tends to infinity, therefore, it is always possible to find a point t_n, belonging to S, within any interval, how-so-ever small, enclosing t, by taking n sufficiently large. Thus t is a limiting point of S.

If t belongs to class (ii), the point

$$t_n = \frac{0}{3} + \frac{a_2}{3.5} + \frac{a_3}{3^2.5} + \cdots + \frac{a'_{2n}}{3^n.5^n} + \frac{a_{2n+1}}{3^{n+1}.5^n} + \cdots$$

is a point of S if

$$a'_{2n} = (a_{2n} + 2),$$

and, then reasoning as before, we can prove that t is a limiting point of S.

Therefore, the set S is dense-in-itself.

That S is a closed set follows from the continuity of $\theta(t)$, for if a point t_r be a limiting point of the zeros of $\theta(t)$, $\theta(t_r)$ must be zero, so that t_r belongs to S.

Thus the set S is perfect.

(c) *S has zero measure.* In order to find the measure of S, we find the sum of the intervals complimentary to S. It is easy to see that there is no point of S in the interval

$$\left(\frac{1}{3}, 1\right),$$

whose length is $\frac{2}{3}$. Then there are no points of S in the five intervals

$$\left\{\frac{1}{3^2.5}, \frac{1}{3.5}\right\}, \left\{\frac{1}{3.5}, \frac{1}{3.5}+\frac{2}{3^2.5}\right\}, \left\{\frac{2}{3.5}+\frac{1}{3^2.5}, \frac{3}{3^2.5}\right\},$$

$$\left\{\frac{3}{3.5}, \frac{3}{3.5}+\frac{2}{3^2.5}\right\}, \left\{\frac{4}{3.5}+\frac{1}{3^2.5}, \frac{1}{3}\right\},$$

The sum of the lengths of these five intervals is $5.\dfrac{2}{3^2.5}=\dfrac{2}{3^2}$.

Then there are 5^2 intervals, each of length $\dfrac{2}{3^3.5^2}$, which do not contain points of the set. The sum of these is $5^2.\dfrac{2}{3^3.5^2}=\dfrac{2}{3^3}$; and so on.

Thus the sum of the intervals which do not contain points of the set S is

$$\frac{2}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\ldots+\frac{2}{3^n}+\ldots=1.$$

Therefore, the measure of S is zero.

The Roots of the equation $\theta(t)=c$.

5. Let the constant c have the following representation when expressed in the scale of 3 :

$$c\equiv\frac{c_1}{3}+\frac{c_2}{3^2}+\frac{c_3}{3^3}+\ldots\ldots+\frac{c_n}{3^n}+\ldots\ldots$$

Then the a's in the representation of

$$t_c\equiv\frac{a_1}{3}+\frac{a_2}{3.5}+\frac{a_3}{3^2.5}+\frac{a_4}{3^2.5^2}+\ldots\ldots\ldots$$

which corresponds to c, satisfy the following conditions :

$$c_1=a_1,$$

and if $c_{n+1}=0$, then $a_{2n+1}=0$ or 2 according as $a_2+a_4+\ldots+a_{2n}$ is even or odd,

or if $c_{n+1}=2$, then $a_{2n+1}=2$ or 0 according as $a_2+a_4+\ldots+a_{2n}$ is even or odd,

or if $c_{n+1}=1$, then $a_{2n+1}=1$, whatever $a_2+a_4+\ldots+a_{2n}$ may be.

It is obvious that there are an infinite number of t_c's, the a's in whose representation satisfy the conditions given above. The points t_c form a set S_c, which is the set of the roots of $\theta(t)=c$, and which we now propose to study.

(*a*) S_c *is unenumerable.* The unenumerability of S_c can be proved by a method similar to the one employed for proving the unenumerability of the set of the zeros. It also follows from the fact that S_c is a perfect set. This we prove below.

(*b*). S_c *is perfect.* Let

$$t_c = \frac{a_1}{3} + \frac{a_2}{3.5} + \frac{a_3}{3^2.5} + \dots + \frac{a_{2n}}{3^n.5^n} + \dots\dots$$

be a root of the equation $\theta(t) = c$.

Then, the point

$$t'_c = \frac{a_1}{3} + \frac{a_2}{3.5} + \dots + \frac{a'_{2n}}{3^n.5^n} + \frac{a_{2n+1}}{3^{n+1}.5^n} + \dots$$

which differs from t_c at the $2n$th place only, is also a root of $\theta(t) = c$, if

$$a_{2n} - a'_{2n} = 2,$$

and therefore, as in §4 (*b*), it follows that t_c is a limiting point of the set S_c. *Thus the set S_c is dense-in-itself.*

That the set S_c is *closed* follows from the continuity of $\theta(t)$.

Therefore, the set S_c is perfect.

(*c*) S_c *has zero measure.*

In the representation of c, if $c_1 = 0$, then there are 2 intervals,

$$\{ \tfrac{1}{3}, \tfrac{2}{3} \} \quad \text{and} \quad \{ \tfrac{2}{3}, 1 \},$$

in each of which there is no point of S_c; if $c_1 = 1$, then there are 2 intervals

$$\{ 0, \tfrac{1}{3} \} \quad \text{and} \quad \{ \tfrac{2}{3}, 1 \}$$

in each of which there is no point of S_c; and similarly if $c_1 = 2$, we find that there are 2 intervals

$$\{ 0, \tfrac{1}{3} \} \quad \text{and} \quad \{ \tfrac{1}{3}, \tfrac{2}{3} \}$$

in each of which there is no point of S_c.

We thus find that, *whatever c_1 may be*, there are 2 intervals, each of length $\frac{1}{3}$, which do not contain points of S_c.

Again, *whatever* c_1 *and* c_2 *may be,* we find that there are 2.5 intervals,* each of length $\frac{1}{3^2.5}$, which do not contain points of S_c, and similarly, whatever c_1, c_2 and c_3 may be, there are 2.5^2 intervals, each of length $\frac{1}{3^3 5.^2}$, which do not contain points of S_c, and so on.

Thus the sum of the intervals that do not contain po intsof S_c is

$$\frac{2}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\ldots\ldots$$

And, therefore, S_c *has zero measure.*

We have thus shown that the roots of the equation $\theta(t)=c$, form a set S_c which is in all respects similar to the set of the zeros of $\theta(t)$.

In my paper "On Infinite Derivates" published in this *Bulletin*, Vol. 16, pp. 79, I have given an example of a continuous function $f(x)$ whose zeros form a set of positive measure.† The function $f(x)$ is, however, not non-differentiable. The method of construction employed by me can be easily modified to give an example of a non-differentiable function $f(x)$ whose zeros form a set of positive measure. Thus *the zeros of a continuous non-differentiable function may form a set of positive measure.*

* For example, if $c_1=1$ and $c_2=0$, the 2.5 intervals are :

$$\left\{\frac{1}{3}+\frac{1}{3^2.5},\ \frac{1}{3}+\frac{2}{3^2.5}\right\},\ \left\{\frac{1}{3}+\frac{2}{3^2.5},\ \frac{1}{3}+\frac{1}{3.5}\right\},\ \left\{\frac{1}{3}+\frac{1}{3.5},\ \frac{1}{3}+\frac{1}{3.5}+\frac{1}{3^2.5}\right\}$$

$$\left\{\frac{1}{3}+\frac{1}{3.5}+\frac{1}{3^2.5},\ \frac{1}{3}+\frac{1}{3.5}+\frac{2}{3^2.5}\right\},\ \left\{\frac{1}{3}+\frac{2}{3.}+\frac{1}{3^2.5},\ \frac{1}{3}+\frac{2}{3.5}+\frac{2}{3^2.5}\right\}$$

$$\left\{\frac{1}{3}+\frac{2}{3.5}+\frac{2}{3^2.5},\ \frac{1}{3}+\frac{3}{3.5}\right\},\qquad \left\{\frac{1}{3}+\frac{3}{3.5},\ \frac{1}{3}+\frac{3}{3.5}+\frac{1}{3^2.5}\right\},$$

$$\left\{\frac{1}{3}+\frac{3}{3.5}+\frac{1}{3^2.5},\ \frac{1}{3}+\frac{3}{3.5}+\frac{2}{3^2.5}\right\},\ \left\{\frac{1}{3}+\frac{4}{3.5}+\frac{1}{3^2.5},\ \frac{1}{3}+\frac{4}{3.5}+\frac{2}{3^2.5}\right\},$$

$$\left\{\frac{1}{3}+\frac{4}{3.5}+\frac{2}{3^2.5},\ \frac{2}{3}\right\}.$$

† The zeros of Volterra's example of a continuous function [whose derivative, though bounded, is not integrable (R)] also from a set of positive measure. *Giorn. di Battaglini*, Vol. XIX, p. 335) ; *cf.* Hobson, *Theory of Functions, etc.*, Vol. I (3rd ed.), p. 490.

3

Did the Babylonians and the Mayas of Central America possess Place-Value Arithmetical Notations?

BY

Saradakanta Ganguli (*Cuttack*).

We are told that the Babylonians and the Mayas of Central America had had systems of arithmetical notation depending on the principle of place-value before the modern place-value decimal notation was invented by the Hindus. The Babylonians are said to have had a sexagesimal system in which the main unit was 60 and the unit 10 occupied a subordinate place. The Mayas are said to have had a vigesimal system.

Regarding the Babylonian sexagesimal system Prof. T. E. Peet writes:* "In a system with a main unit 60 and a subsidiary unit 10 we should have to represent each portion by two digits, thus 32.12.43 would stand for (32×60^2) plus (12×60) plus (43×1), or 115, 963. This is precisely what the Sumerians (*i.e.*, a section of the Babylonians) did, and it was here that their secondary unit 10 served them in such good stead. In the mathematical tablets two signs alone are used in this notation, the sign for 1 and the sign for 10. The number above referred to would be as follows:—three tens and two units; a ten and two units; four tens and three units: the fact that the first group is to be multiplied by 60^2, the second by 60, and the last by 1 is taken for granted, just as the multiplication of 3, 6 and 5 by 100, 10 and 1 respectively is in our decimal notation for 365."

Prof. Peet then adds: "The system was still further perfected by the use of a sign for zero in cases where the middle term was missing, *e.g.* in 12.0.33 which stood for (12×60^2) plus (0×60) plus (33×1). Here we have all the elements of positional notation with one exception: there was nothing to correspond to our decimal point. * * * When, however, the Sumerian wrote 12.25.33 he had no means of showing whether the lowest unit, that to be multiplied by 33 was 3600, 60, 1, $\frac{1}{60}$ or some other sexagesimal unit. All that was fixed

* *The Rhind Mathematical Papyrus*, 1923, p. 28.

was that whatever the lowest of these units (to be multiplied by 33) was, the next (to be multiplied by 25) was 60 times greater, and the highest (to be multiplied by 12) 3600 times as great."

Dr. Florian Cajori writes :* "Perhaps five or six centuries † before the Hindus gave a systematic exposition of their *decimal* number system with its zero and principle of local value, the Maya in the flatlands of Central America had evolved systematically a *vigesimal* number system employing a zero and the principle of local value. In the Maya number system found in the codices the ratio of increase of successive units was not 10, as in the Hindu system; it was 20 in all positions except the third. That is, 20 units of the lowest order (*kins*, or days) make one unit of the next higher order (*uinals*, or 20 days), 18 uinals make one unit of the third order (*tun*, or 360 days), 20 tuns make one unit of the fourth order (*katun*, or 7200 days), 20 katuns make one unit of the fifth order (*cycle*, or 144,000 days) and finally 20 cycles make 1 *great cycle* of 2,880,000 days."

In spite of the above statements having been made by acknowledged authorities on the history of mathematics the present writer finds it difficult to accept them unreservedly for the following reasons :

(1) Both the Babyloyians ‡ and the Mayas § had previously decimal systems of notation which they are said to have subsequently given up. But the rest of the world does not supply us with another instance of a people who have abandoned a decimal system of notation in favour of a natural or unnatural system, although there are instances of peoples who have adopted decimal systems of notation in supersession of the previously existing other natural systems.¶

(2) Notation follows and does not precede numeration. The scale of notation is, therefore, the scale of numeration. Previous employment of decimal systems of notation by the Babylonians and the Mayas

* *A History of Mathematics*, 1922, p. 69.

† This interval should be 'one or two centuries.' For, the Hindus gave an exposition of the modern decimal notation towards the end of the 5th century A. D., and not in the 9th century as supposed by Dr. Cajori. For Āryabhaṭa's exposition of the modern notation the reader is referred to the present writer's article "The elder Āryabhaṭa and the modern arithmetical notation" published in the *American Mathematical Monthly* for October, 1927.

‡ Sir Thomas Heath, *A History of Greek Mathematics*, Vol. I, p. 28 : Dr. F. Cajori, *A History of Mathematics* (1922), p. 4.

§ Cajori, *A History of Mathematics* (1922), p. 69.

¶ Peacock, *Arithmetic* in the Encyclopædia of Pure Mathematics (1847), pp. 371 and 385.

prove that they had a decimal scale of numeration and that their numerical language was adapted to that scale. Once the numerical language is fixed, any alteration in the scale of notation is attended with tremendous difficulties except in the case of small numbers which do not require the use of new units of the third or higher order. Although we are familiar with sexagesimal units, we find it extremely difficult to form any idea of the number 5.49.37 (expressed in the sexagesimal scale) without first expressing it in our decimal scale. But we can easily form an idea of a period of 5 hours 49 minutes 37 seconds, though, when the same period is stated as 1082977 seconds, we cannot make an idea of it without first reducing it to sexagesimal units of time. In the case of numbers our language is adapted to the decimal scale; hence, numbers expressed in any other scale present difficulties. In the case of time our language is adapted to the sexagesimal scale; hence, time, even when expressed in our familiar decimal scale, can make no impression on us. How could the Babylonians and the Mayas, with numerical languages adapted to the previously existing decimal scale, easily understand and use numbers expressed in a different scale? In order to adopt a new scale of notation did they abandon their old numerical languages? Can we understand the people to have acquiesced in a change that must have resulted in so much confusion and inconvenience to them? Does the rest of the world supply another instance of such a change in numerical language?

(3) The sexagesimal system of notation is not a natural one. Why then did the Babylonians almost abandon a natural system of notation (namely, the decimal system) in favour of an unnatural one? Sir Thomas Heath presumes* that the Babylonian authors of the supposed sexagesimal system were fully alive to the convenience of 60 as a base with so many divisors, combining as it does the advantages of 12 and 10. With due deference to the opinion of Sir Thomas Heath the present writer begs to differ from him. For, the perception of the advantages of the sexagesimal system of notation presupposes an advanced state of arithmetical knowledge which the Babylonians did not certainly possess at that time. A people who could not perceive the necessity of indicating the order of the sexagesimal unit by which the lowest term of a number expressed in their alleged notation is to be multiplied, can hardly be credited with an advanced state of arithmetical knowledge. The mathematicians of modern times are not less alive to the advantages of the sexagesimal or duodecimal system of notation. Yet they

* *A Histroy of Greek Mathematics*, Vol. I, p. 29.

have not abandoned the decimal system of notation or given it a subordinate place in their calculations.

Dr. Cajori's account of the Maya number system, quoted above, is sufficient to show that the object of the system was not to supersede the existing decimal system of notation by a vigesimal one but probably to coin units of time, as indicated by the terms, *kin* (day), *uinal* (20 days), *tun* (360 days), *katun* (7200 days), etc. If the Maya vigesimal system were a system of arithmetical notation, the ratio of increase of successive units would be 20 throughout and not 18 in the third position. The Maya probably reckoned 360 days to the year and was, therefore, obliged to introduce a unit of time which was 18 times as great as the next lower unit.

(4) We also use sexagesimal units in expressing short intervals of time, *e.g.*, 5 hours 0 minutes 45 seconds or hours 5.0.45. We also use a vigesimal unit in expressing a sum of money in English coins, *e.g.*, 5 pounds 0 shillings or £ 5.0. But we cannot, therefore, claim that besides our decimal system of notation we also use sexagesimal and vigesimal systems of notation depending on the principle of local value and the employment of zero. Did not the Babylonians and the Mayas respectively use the sexagesimal and the vigesimal units exactly in the same way as we ourselves use similar units ?

(5) Peacock holds that "the natural scales of notation alone have ever met with general adoption."* He has also given his reasons for thinking that "the preference shown amongst Scandinavian nations for the number *twelve*, and its very general use in the division of concrete numbers, furnish no sufficient ground for considering it as having been used as the radix of a scale of notation, however nearly in some respects it may have approximated to it."† Are not the Babylonian use of the number *sixty* and the Maya use of the number *twenty* similar to the use of the number *twelve* amongst the Scandinavian nations ?

From the above considerations the present writer thinks that the Tables of Senkereh and the Maya codices which are supposed to bear testimony to the alleged employment of the sexagesimal and vigesimal systems of notation by the Babylonians and the Mayas respectively should be re-studied and re-interpreted.

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* *The Encyclopædia of Pure Mathematics* (1847), p. 371.
† *Ibid*, p. 392.

4

On the Limiting Points of the Zeros of a Non-differentiable Function first given by Dini

BY

Bholanath Mukhopadhyay

(*Calcutta*).

The publication of Prof. G. Prasad's remarkable paper * on the zeros of Weierstrass's non-differentiable function has naturally led mathematicians to study the zeros of different types of non-differentiable functions.

The object of the present paper is chiefly to ascertain whether in a finite interval, say, (0, 1) there is any limiting point of the zeros of the non-differentiable function

$$F(x) = \sum_{1}^{\infty} \sin \frac{(16^n \pi x)}{2^n}$$

and as also of the general non-differentiable function

$$f(x) = \sum_{1}^{\infty} a^n \sin b^n \pi x$$

where $0<a<1$ and b is an *even* integer and $\quad ab > 1+\frac{3\pi}{2}$.

The results obtained by me are all new.

§ 1.

1. It easily follows that

$$F\left(\frac{p}{16}\right)=0, \text{ where } p=0, 1, 2, \ldots\ldots 15, 16.$$

Hence, $\frac{p}{16}$, $(p=0, 1, 2, \ldots\ldots 15, 16)$ gives roots of $F(x)=0$.

* *Proceedings of the Benares Math. Society*, Vol. XI, pp. 1-8.

2. (a) $F\left(\frac{1}{16^k}\right)=\frac{1}{2}\sin\frac{\pi}{16^{k-1}}+\frac{1}{2^2}\sin\frac{\pi}{16^{k-2}}\ldots+\frac{1}{2^{k-1}}\sin\frac{\pi}{16}$

(k being any positive integer greater than one).

Therefore $F\left(\frac{1}{16^k}\right)>0$, for every positive integral value of k greater than one.

(b) Again,

$$F\left(\frac{3/2}{16^k}\right)=\frac{1}{2}\sin\frac{3\pi/2}{16^{k-1}}+\frac{1}{2^2}\sin\frac{3\pi/2}{16^{k-2}}+\ldots+\frac{1}{2^{k-1}}\sin\frac{3\pi/2}{16}-\frac{1}{2^k},$$

(k being any positive integer).

Now, $\frac{1}{2}\sin\frac{3\pi/2}{16^{k-1}}+\frac{1}{2^2}\sin\frac{3\pi/2}{16^{k-2}}+\ldots+\frac{1}{2^{k-1}}\sin\frac{3\pi/2}{16}$

$$=\frac{1}{2^{k-1}}\left[\sin\frac{3\pi/2}{16}+2\sin\frac{3\pi/2}{16^2}+\ldots\ldots+2^{k-2}\sin\frac{3\pi/2}{16^{k-1}}\right].$$

$$=\frac{1}{2^{k-1}}\left[\sin\alpha+2\sin\frac{\alpha}{16}+2^2\sin\frac{\alpha}{16^2}+\ldots\ldots+2^{k-2}\sin\frac{\alpha}{16^{k-2}}\right]$$

(where $\alpha=\frac{\frac{3\pi}{2}}{16}$.)

$$<\frac{1}{2^{k-1}}\left[\alpha+2.\frac{\alpha}{16}+2^2.\frac{\alpha}{16^2}+\ldots\ldots+2^{k-2}\frac{\alpha}{16^{k-2}}\right].$$

$$<\frac{\alpha}{2^{k-1}}.\left[1+\frac{1}{8}+\frac{1}{8^2}+\ldots\ldots+\frac{1}{8^{k-2}}\right]$$

$$<\frac{\alpha}{2^{k-1}}\left[1+\frac{1}{8}+\frac{1}{8^2}+\ldots\ldots\text{ up to }\infty\right]$$

$$<\frac{\alpha}{2^{k-1}}.\frac{8}{7}.$$

i.e. $< \frac{1}{2^k} \cdot \frac{33}{49}$.

Evidently,

$$\frac{1}{2^k} > \frac{1}{2^k} \cdot \frac{33}{49}.$$

Therefore $F\left(\frac{3/2}{16^k}\right) < 0$, for every positive integral value of k.

(c) Hence, it follows that there is at least one root of $F(x)=0$ in each of the intervals

$$\left(\frac{1}{16^2}, \frac{3/2}{16^2}\right), \left(\frac{1}{16^3}, \frac{3/2}{16^3}\right), \ldots \left(\frac{1}{16^k}, \frac{3/2}{16^k}\right) \ldots$$

and also in each of the intervals.

$$\left(\frac{3/2}{16^3}, \frac{1}{16^2}\right), \left(\frac{3/2}{16^4}, \frac{1}{16^3}\right), \ldots \left(\frac{3/2}{16^{k+1}}, \frac{1}{16^k}\right) \ldots$$

(k being any positive integer greater than one).

Thus, 0 is a limiting point of a set of zeros of $F(x)$.

3. The roots of $F(x)=0$, lying in the first set of intervals, given above, may be conveniently represented by the general expression

$$\frac{1}{16^k}\left(1 + \frac{1}{2}\lambda\right)$$

where $\frac{1}{8} < \lambda < \frac{5}{8}$ and k is any positive integer greater than one.

Proof :

Any root lying between $\frac{1}{16^k}$ and $\frac{3/2}{16^k}$ is represented by

$$\frac{1}{16^k} + \frac{1}{2}\lambda.\ \frac{1}{16^k}, \text{ where } 0 < \lambda < 1.$$

Now it can be easily proved that

$$\frac{1}{8} < \lambda < \frac{5}{8}.$$

For, $$\frac{1}{16^k}\left(1+\frac{1}{2}\cdot\frac{1}{8}\right)=\frac{1}{16^k}+\frac{1}{16^{k+1}}=\frac{17}{16^{k+1}};$$

and, $$\frac{1}{16^k}\left(1+\frac{1}{2}\cdot\frac{5}{8}\right)=\frac{1}{16^k}+\frac{5}{16^{k+1}}=\frac{21}{16^{k+1}};$$

and as shown below, $$F\left(\frac{17}{16^{k+1}}\right)>0, \text{ for } k>1$$

and $$F\left(\frac{21}{16^{k+1}}\right)<0, \text{ for } k>1.$$

(a) $$F\left(\frac{17}{16^{k+1}}\right)=\frac{1}{2}\sin\frac{17\pi}{16^k}+\frac{1}{2^2}\sin\frac{17\pi}{16^{k-1}}+\ldots+\frac{1}{2^k}\sin\frac{17\pi}{16}$$

$$=\frac{1}{2}\sin\frac{17\pi}{16^k}+\frac{1}{2^2}\sin\frac{17\pi}{16^{k-1}}+\ldots+\frac{1}{2^{k-1}}\sin\frac{17\pi}{16^2}-\frac{1}{2^k}\sin\frac{\pi}{16}.$$

Now, $$\frac{1}{2}\sin\frac{17\pi}{16^k}+\frac{1}{2^2}\sin\frac{17\pi}{16^{k-1}}+\ldots+\frac{1}{2^{k-1}}\sin\frac{17\pi}{16^2}$$

$$>\frac{1}{2}\cdot\frac{17\pi}{16^k}\cdot\frac{2}{\pi}+\frac{1}{2^2}\cdot\frac{17\pi}{16^{k-1}}\cdot\frac{2}{\pi}+\ldots+\frac{1}{2^{k-1}}\frac{17\pi}{16^2}\cdot\frac{2}{\pi}$$

$$>\frac{2.17}{2^{k-1}}.\frac{1}{16^2}\left[1+\frac{1}{8}+\left(\frac{1}{8}\right)^2+\ldots+\left(\frac{1}{8}\right)^{k-2}\right]$$

$$>\frac{2.17}{2^{k-1}}\cdot\frac{1}{16^2}\cdot\frac{8}{7}\left\{1-\frac{1}{8^{k-1}}\right\}$$

i.e., $$>\frac{17}{2^{k-1}}\cdot\frac{1}{7.16}\left\{1-\frac{1}{8^{k-1}}\right\}.$$

And, $$\frac{1}{2^k}\sin\frac{\pi}{16}<\frac{1}{2^k}\cdot\frac{\pi}{16}$$

Now, $$\frac{17}{2^{k-1}}\cdot\frac{1}{7.16}\cdot\left\{1-\frac{1}{8^{k-1}}\right\}-\frac{1}{2^k}\cdot\frac{\pi}{16}$$

$$=\frac{1}{2^k}\cdot\frac{1}{16.7}\left\{34\left(1-\frac{1}{8^{k-1}}\right)-22\right\}$$

$$=\frac{1}{2^k}\cdot\frac{1}{16.7}\left\{12-\frac{34}{8^{k-1}}\right\}$$

$$>0, \text{ if } k>1.$$

Thus it is proved that $F\left(\frac{17}{16^{k+1}}\right)>0$, for $k>1$.

(b)

$$F\left(\frac{21}{16^{k+1}}\right)=\frac{1}{2}\sin\frac{21\pi}{16^k}+\frac{1}{2^2}\sin\frac{21\pi}{16^{k-1}}+\dots+\frac{1}{2^k}\sin\frac{21\pi}{16}$$

$$=\frac{1}{2}\sin\frac{21\pi}{16^k}+\frac{1}{2^2}\sin\frac{21\pi}{16^{k-1}}+\dots+\frac{1}{2^{k-1}}\sin\frac{21\pi}{16^2}-\frac{1}{2^k}\sin\frac{5\pi}{16}.$$

Now, $$\frac{1}{2}\sin\frac{21\pi}{16^k}+\frac{1}{2^2}\sin\frac{21\pi}{16^{k-1}}+\dots+\frac{1}{2^{k-1}}\sin\frac{21\pi}{16^2}$$

$$<\frac{1}{2}\cdot\frac{21\pi}{16^k}+\frac{1}{2^2}\cdot\frac{21\pi}{16^{k-1}}+\dots+\frac{1}{2^{k-1}}\cdot\frac{21\pi}{16^2}$$

$$<\frac{21\pi}{2^{k-1}}\cdot\frac{1}{16^2}\left[1+\frac{1}{8}+\frac{1}{8^2}+\dots+\frac{1}{8^{k-2}}\right]$$

$$<\frac{21\pi}{2^{k-1}}\cdot\frac{1}{16^2}\left[1+\frac{1}{8}+\frac{1}{8^2}+\dots\text{ up to }\infty\right]$$

$$<\frac{21\pi}{2^{k-1}}\cdot\frac{1}{16^2}\cdot\frac{8}{7}.$$

i.e., $$<\frac{1}{2^{k+3}}\cdot\frac{33}{7}.$$

And, $$\frac{1}{2^k}\sin\frac{5\pi}{16} > \frac{1}{2^k}\cdot\frac{5\pi}{16}\cdot\frac{2}{\pi}$$

i.e., $$> \frac{5}{2^{k+3}}.$$

But, $$\frac{5}{2^{k+3}} > \frac{1}{2^{k+3}}\cdot\frac{33}{7}.$$

Therefore $$F\left(\frac{21}{16^{k+1}}\right) < 0.$$

Similarly, the roots of $F(x)=0$ lying in the second set of intervals, given in Art. 2, may be represented by the general expression

$$\frac{3/2}{16^{k+1}}\left(1+\mu.\frac{29}{3}\right)$$

where $0<\mu<1$ and k is any positive integer greater than one.

It is also clear that both λ and μ can be approximated to as closely as we please.

§ 2.

4. Now, I proceed to prove the existence of some other limiting points.

$$F\left(\frac{2}{16}+\frac{1}{16^k}\right)=\frac{1}{2}\sin\frac{\pi}{16^{k-1}}+\frac{1}{2^2}\sin\frac{\pi}{16^{k-2}}+\ldots+\frac{1}{2^{k-1}}\sin\frac{\pi}{16}$$

$$>0, \text{ as in Art. 2 } (a)$$

(k being any positive integer greater than one).

Also,

$$F\left(\frac{2}{16}+\frac{3/2}{16^k}\right)=\frac{1}{2}\sin\frac{3\pi/2}{16^{k-1}}+\frac{1}{2^2}\sin\frac{3\pi/2}{16^{k-2}}+\ldots$$

$$+\frac{1}{2^{k-1}}\sin\frac{3\pi/2}{16}-\frac{1}{2^k}$$

Therefore $F\left(\frac{2}{16}+\frac{3/2}{16^k}\right) < 0$, as in Art. 2 (b).

Similarly, it can be shewn that

$$F\left(\frac{m}{16}+\frac{1}{16^k}\right) > 0,$$

$$F\left(\frac{m}{16}+\frac{3/2}{16^k}\right) < 0,$$

$$F\left(\frac{m}{16}-\frac{1}{16^k}\right) < 0,$$

$$F\left(\frac{m}{16}+\frac{3/2}{16^k}\right) > 0,$$

where m is any positive even integer less than sixteen and k is any positive integer greater than one

Hence, it follows that there are roots of $F(x)=0$ in any arbitary neighbourhood of each of the points $\frac{m}{16}$ (m being any positive even integer less than 16).

Therefore, $\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}$

are limiting points of the sets of zeros of $F(x)$.

5. Again,

$$F\left(\frac{1}{16}+\frac{1}{16^k}\right) = -\frac{1}{2}\sin\frac{\pi}{16^{k-1}} + \frac{1}{2^2}\sin\frac{\pi}{16^{k-2}} + \dots + \frac{1}{2^{k-1}}\sin\frac{\pi}{16}$$

$$= \frac{1}{2^{k-1}}\left[\sin\frac{\pi}{16} + 2\sin\frac{\pi}{16^2} + \dots + 2^{k-2}\sin\frac{\pi}{16^{k-1}}\right] - \sin\frac{\pi}{16^{k-1}}$$

The above expression within brackets

$$= \sin\alpha + 2\sin\frac{\alpha}{16} + \dots + 2^{k-2}\sin\frac{\alpha}{16^{k-2}}$$

$$\left(\text{where } \alpha = \frac{\pi}{16}\right)$$

$$> \frac{2}{\pi}\alpha + 2.\frac{2}{\pi}.\frac{\alpha}{16} + \dots + 2^{k-2}.\frac{2}{\pi}.\frac{\alpha}{16^{k-2}}$$

$$>\frac{2}{\pi}\,\alpha\left\{1+\left(\frac{1}{8}\right)+\left(\frac{1}{8}\right)^2+\ldots+\left(\frac{1}{8}\right)^{k-2}\right\}$$

$$>\frac{2}{\pi}.\ \alpha.\ \frac{8}{7}.\left\{1-\left(\frac{1}{8}\right)^{k-1}\right\}$$

$$i.e.,\ >\frac{1}{7}.\left\{1-\left(\frac{1}{8}\right)^{k-1}\right\}$$

Now, $\sin\dfrac{\pi}{16^{k-1}}<\dfrac{\pi}{16^{k-1}}$.

Therefore, $F\left(\dfrac{1}{16}+\dfrac{1}{16^k}\right)>0$

if, $\dfrac{1}{2^{k-1}}.\dfrac{1}{7}\left\{1-\left(\dfrac{1}{8}\right)^{k-1}\right\}>\dfrac{\pi}{16^{k-1}}$

i.e., if $8^{k-1}>23$,

which is true for any positive integral value of k greater than 2.

Therefore $F\left(\dfrac{1}{16}+\dfrac{1}{16^k}\right)>0$, for $k>2$.

Also,

$$F\left(\frac{1}{16}+\frac{3/2}{16^k}\right)=-\frac{1}{2}\sin\frac{3\pi/2}{16^{k-1}}+\frac{1}{2^2}\sin\frac{3\pi/3}{16^{k-2}}+\ldots$$

$$+\frac{1}{2^{k-1}}\sin\frac{3\pi/2}{16}-\frac{1}{2^k},$$

Now, it has been shewn in Art. 2(*b*) that

$$\left[\frac{1}{2}\sin\frac{3\pi/2}{16^{k-1}}+\frac{1}{2^2}\sin\frac{3\pi/2}{16^{2-1}}+\ldots+\frac{1}{2^{k-1}}\sin\frac{3\pi/2}{16}-\frac{1}{2^k}\right]$$

is negative, therefore

$$F\left(\frac{1}{16}+\frac{3/2}{16^k}\right)<0.$$

Also it easily follows that

$$F\left(\frac{1}{16}-\frac{1}{16^k}\right)<0; \qquad \text{for } k>2;$$

and $$F\left(\frac{1}{16}-\frac{3/2}{16^k}\right);>0.$$

Similarly, it can be shewn that

$$F\left(\frac{q}{16}+\frac{1}{16^k}\right)>0,$$

$$F\left(\frac{q}{16}+\frac{3/2}{16^k}\right)<0,$$

$$F\left(\frac{q}{16}-\frac{1}{16^k}\right)<0,$$

$$F\left(\frac{q}{16}-\frac{3/2}{16^k}\right)<0,$$

where q is any positive odd integer less than 16 and k is any positive integer greater than 2.

Hence, it follows that there are roots of $F(x)=0$ in every neighbourhood, however small, of each of the points given by $\frac{q}{16}$, where q is a positive odd integer less than 16.

Therefore, $\frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}, \frac{9}{16}, \frac{13}{16}, \frac{15}{16}, \frac{17}{16}$ are limiting points of the sets of zeros of $F(x)$.

6. Also, it easily follows that

$$F\left(1-\frac{1}{16^k}\right)=-\left[\frac{1}{2}\sin\frac{\pi}{16^{k-1}}+\frac{1}{2^2}\sin\frac{\pi}{16^{k-2}}+\dots\right.$$
$$\left.+\frac{1}{2^{k-1}}\sin\frac{\pi}{16}\right]$$

Therefore $$F\left(1-\frac{1}{16^k}\right)<0.$$

Again, $F\left(1-\frac{3/2}{16^k}\right) = -\left[\frac{1}{2}\sin\frac{3\pi/2}{16^{k-1}} + \frac{1}{2^2}\sin\frac{3\pi/2}{16^{k-2}} \ldots\ldots\right.$

$$\left. + \frac{1}{2^{k-1}}\sin\frac{3\pi/2}{16} - \frac{1}{2^k}\right].$$

Therefore $F\left(1-\frac{3/2}{16^k}\right) > 0$ (See Art. 2(b)

Hence, 1 is also a limiting point of a set of zeros of $F(x)$.

§ 3.

7. The limiting points of the zeros of the function $f(x) = \sum_1^\infty a^n \sin(b^n\pi x)$ can also be investigated in the same way as in the case of the function $F(x)$.

$$f(x) = a\sin b\pi x + a^2\sin b^2\pi x + a^3\sin b^3\pi x + \ldots\ldots$$

$$f\left(\frac{1}{b^k}\right) = a\sin\frac{\pi}{b^{k-1}} + a\sin\frac{\pi}{b^{k-2}} + \ldots + a^{k-}\sin\frac{\pi}{b}.$$

Therefore $f\left(\frac{1}{b^k}\right) > 0$, for $k > 1$.

Again,

$$f\left(\frac{\frac{3}{2}}{b^k}\right) = \sin\frac{3\pi/2}{b^{k-1}} + a^2\sin\frac{3\pi/2}{b^{k-2}} + \ldots\ldots + a^{k-1}\sin\frac{3\pi/2}{b} - a^k$$

Now,

$$a\sin\frac{3\pi/2}{b^{k-1}} + a^2\sin\frac{3\pi/2}{b^{k-1}} + a^3\sin\frac{3\pi/2}{b^{k-2}} + \ldots\ldots + a^{k-1}\sin\frac{3\pi/2}{b}$$

$$= a^{k-1}\left[\sin\frac{3\pi/2}{b} + \frac{1}{a}\sin\frac{3\pi/2}{b^2} + \frac{1}{a^2}\sin\frac{3\pi/2}{b^3} + \ldots\ldots\right.$$

$$\left. + \frac{1}{a^{k-2}}\sin\frac{3\pi/2}{b^{k-1}}\right]$$

$$=a^{k-1}\left[\sin \alpha+\frac{1}{a}\sin\frac{\alpha}{b}+\frac{1}{a^2}\sin\frac{\alpha}{b^2}+\ldots+\frac{1}{a^{k-2}}\sin\frac{\alpha}{b^{k-2}}\right]$$

$$\left(\text{where } \alpha=\frac{3\pi/2}{b}\right).$$

$$< a^{k-1}\left[\alpha+\frac{\alpha}{ab}+\frac{\alpha}{a^2b^2}+\ldots+\frac{\alpha}{(ab)^{k-2}}\right]$$

$$< a^{k-1}.\alpha\left[1+\frac{1}{ab}+\frac{1}{a^2b^2}+\ldots+\frac{1}{(ab)^{k-2}}\right]$$

$$< a^{k-1}.\alpha\left[1+\frac{1}{ab}+\frac{1}{a^2b^2}+\ldots\ldots\text{up to }\infty\right]$$

$$< a^{k-1}.\alpha.\frac{1}{1-\dfrac{1}{ab}}.$$

$$i.e., < a^k\,\frac{3\pi}{2}.\frac{1}{ab-1}.$$

$$\text{Now, } a^k-\left(a^k.\ \frac{3\pi}{2}.\ \frac{1}{ab-1}\right).$$

$$=a^k\left\{1-\frac{3\pi}{2}.\frac{1}{ab-1}\right\}$$

$$=a^k\left[\frac{(ab-1)-\dfrac{3\pi}{2}}{(ab-1)}\right]$$

$$=\text{positive, since } ab > 1+\frac{3\pi}{2}$$

$$\text{Therefore } f\left(\frac{\frac{3}{2}}{b^k}\right) < 0.$$

Thus, there are roots of $f(x)=0$ in every neighbour-hood, however small, on the right of 0.

Hence, 0 is a limiting point (on the right) of the zeros of $f(x)$

8. $f\left(\frac{p}{b}\right)=0$, where $p=0, 1, 2, 3, \ldots b-1, b$.

And, it can be shewn precisely in the same manner as it has been done in the case of $F(x)$, that each of the points given by $\frac{p}{b}$, $(p=0, 1, 2, \ldots b-1, b)$, is a limiting point of a set of zeros of $f(x)$.

My best thanks are due to Prof. Ganesh Prasad for his encouragement and interest.

5

"Aryabhata's Lost Work"

BY

Probodhchandra Sengupta.

(*Calcutta*)

1. In his *Brahmasphuta Siddhanta*, Chapter XI, 5, Brahmagupta speaks of Aryabhata's two works in the following way :—

"As in both the works the number of sun's revolutions is spoken of as 4320000, their planetary cycle is clear, *i.e.*, of 4320000 years. Why then is there a difference of 300 civil days in the same cycle of the two books?"

Again in stanza 12 of the same chapter he says—

"In 14400 years elapsed of the *Mahayuga* (4320000 years), there is produced a difference of *one* day in counting first from the midnight and then from the sunrise."

In explanation of the last stanza M. M. Sudhakara Dvivedi, most probably on the authority of Chaturveda * the commentator writes as follows :—

"Two works were written by Aryabhata. In one the number of civil days in a *Mahayuga* was given as 1577917500 and the creation of the world spoken of as finished at sunrise at Lanka; in the other work the number of civil days in a *Mahayuga* was given to be 1577917800 and the creation as finished at midnight. In both the works the number of years in a *yuga* was the same, *viz.*, 4320000."

Varahamihira in his Pancha Siddhantika XV, 20 writes :—

"Aryabhata maintains that the beginning of the day is to be reckoned from midnight at Lanka; and the same teacher again says that the day begins from sunrise at Lanka."

2. One of these books is undoubtedly the *Aryabhatiyam*, which was first edited with the commentary of Parameswara by Dr. Kern. A translation into English of this work by the writer of this paper has already appeared in the Calcutta University Journal of Letters, Vol. XVI. The second work has not been discovered yet. At present we

* He lived about 878 of the Christian era.

have † three works which treat of Aryabhata's system of astronomy:—

(1) The *Aryabhatiyam* (499 A.D.).

(2) The *Khandakhadyaka* of Brahmagupta (665 A.D.).

(3) The *Sishyadhibriddhida* of Lalla.

As Lalla wrote a commentary on the *Khandakhadyaka*, he undoubtedly flourished after Brahmagupta. The *Khandakhadyaka* with Amraj's commentary has been edited by Pandit Babua Misra and published by the Calcutta University. In the present paper are set forth the results of my study of *this work of Brahmagupta in which he constructed a much simpler system of astronomical methods which would lead to the same results as those obtained from the work of Aryabhata.* I have no hesitation in saying that the astronomical elements used in this work were mostly taken without any alterations from the lost work of Aryabhata. In this book also the beginning of the astronomical day is placed at midnight and the number of civil days in a *Mahayuga* used, is 1577917800. The various astronomical elements of this work are mostly identical with those given in the *Surya-Siddhanta* of Varahamihira, but they are in many cases different from those of the *Aryabhatiyam.* These are exhibited in the following tabular form:—

3. The Results in a tabular form:—

(1) Planetary revolutions in a *Mahayuga* of 4320000 years.

	According to *Aryabhatiyam.*	According to *Khandakhadyaka.*	According to *Surya-siddhanta* of Varaha.	According to the Modern *Surya-siddhanta.*
Of Moon	57753336	57753336	57753336	57753336
,, Sun	4320000	4320000	4320000	4320000
,, Mars	2296824	2296824	2296824	2296832
,, Jupiter	364224	364220	364220	364220
,, Saturn	146564	146564	146564	146568
,, Moon's apogee.	488219	488219	488219	488203
,, Venus	7022388	7022388	7022388	7022376
,, Mercury	17937020	17937000	17937000	17937060
,, Moon's nodes.	232226	232226	232226	232238

† The *Mahasiddhanta* also professes to describe Aryabhata's system, and is written by *an* Aryabhata who according to my impression, must have lived after Bhaskara.

(2) Longitudes of the apogees of the orbits of planets.

	According to *Aryabhatiyam*.	According to *Khanda-khadyaka*.	According to *Suryasiddhanta* of Varaha.	According to the Modern *Suryasiddhanta*.
Of Sun	78°	80°	80°	77°17′
,, Mercury	210°	220°	220°	&c. have to be calculated from the data given in the text.
,, Venus	90°	80°	80°	
,, Mars	118°	110°	110°	
,, Jupiter	180°	160°	160°	
,, Saturn	236°	240°	240°	

(3) Dimensions of the epicycles of apsis.

	According to *Aryabhatiyam*.	According to *Khanda-khadyaka*.	According to *Suryasiddhanta* of Varaha.	According to the Modern *Suryasiddhanta*.
Of Sun	13°30′	14°	14°	$13\frac{2}{3}$° to 14°
,, Moon	31°30′	31°	31°	$31\frac{2}{3}$° to 32°
,, Mercury	$22\frac{1}{2}$° to $31\frac{1}{2}$°.	28°	28°	28° to 30°
,, Venus	9° to 18°	14°	14°	11° to 12°
,, Mars	63° to 81°	70°	70°	72° to 75°
,, Jupiter	$31\frac{1}{2}$° to $36\frac{1}{2}$°	32°	32°	32° to 33°
,, Saturn	$40\frac{1}{2}$° to $58\frac{1}{2}$°	60°	60°	48° to 49°

(4) Dimensions of the *Sighra* epicycles (*i.e.*, of conjunctions).

	According to *Aryabhatiyam*.	According to *Khanda-khadyaka*.	According to *Suryasiddhanta* of Varaha.	According to the Modern *Suryasiddhanta*.
Of Saturn	$36\frac{1}{2}$° to 40°	40°	40°	39° to 40°
,, Jupiter	$67\frac{1}{2}$° to 72°	72°	72°	70° to 72°
,, Mars	$229\frac{1}{2}$° to $238\frac{1}{2}$°	234°	234°	232° to 235°
,, Venus	$256\frac{1}{2}$° to $265\frac{1}{2}$°	260°	260°	260° to 262°
,, Mercury	$130\frac{1}{2}$° to $139\frac{1}{2}$°	132°	132°	132° to 133°

(5) Longitudes of the nodes of the orbits of planets.

	According to *Aryabhatiyam.*	According to *Khanda-khadyaka.*	According to *Suryasiddhanta* of Varaha.	According to the Modern *Suryasiddhanta.*
Of Mars	40°	40°	Not stated in the text.	Have to be calculated from the data of the text.
,, Mercury	20°	20°		
,, Jupiter	80°	80°		
,, Venus	60°	60°		
,, Saturn	100°	100°		

(6) Orbital inclinations (Geocentric) to the ecliptic.

	According to *Aryabhatiyam.*	According to *Khanda-khadyaka.*	According to *Suryasiddhanta* of Varaha.	According to the Modern *Suryasiddhanta.*
Of Mars	90′	90′	10′	90′
,, Mercury	120′	120′	135′	120′
,, Jupiter	60′	60′	101′	60′
,, Venus	120′	120′	101′	120′
,, Saturn	120′	120′	135′	100′
(7) Number of civil days in a *Mahayuga* of 4320000 years.	1577917500	1577917800	1577917800	1577917828
(8) Beginning of the astronomical day.	Sunrise at Lanka.	Midnight at Lanka.	Midnight at Lanka.	Midnight at Lanka.

Between the *Khandaknadyaka* and the *Suryasiddhanta* of Varaha there is thus agreement in at least 28 or 29 principal elements which are all essential for the calculation of longitudes of planets, and disagreement in only 5 minor elements. Between the *Khandakhadyaka* and the *Aryabhatiyam* there is agreement in 16 elements only, while between the two *Suryasiddhantas* agreement is only in 3 or 4 elements.

4. *Methods of getting at the figures* :—

The section (1) of the planetary revolutions used in the *Khanda-khadyaka* has been deduced from the rules for finding the mean longi-

tudes of planets, as they are not directly stated. The longitudes of apogees and the dimensions of the epicycles of apsis are directly stated. The dimensions of the *sighra* epicycles have been deduced from the tables of *sighra* equations for different values of the *sighra* anomaly. The following are the illustrations.

(1) The sun's mean longitude in A days is given to be $=\frac{800A}{292207}$ revolutions; from which it is seen that the sun's revolutions are 4320000 (*i.e.*, $\equiv 800\times 5400$ (in 1577917800 (*i.e.*, 5400×292207) days.

(2) The moon's mean longitude in A days is given to be $=\frac{600A}{16393}$ revolutions $-\frac{A}{4929}$ minutes; this was equated to $\frac{xA}{1577917800}$, and x worked out to be 57753336.

(3) Mars' table of *sighra* equations for different angles of *sighra* anomalys is given as follows:—

	(*i*)	(*ii*)	(*iii*)		(*iv*)	
Sighra anomaly =	28°	60°	90°	121°	135°	etc.
Corresponding *sighra* Equation =	11°	23°	33°	40°	40°30′	etc.

First the *sighra* periphery was deduced from the sets (*i*), (*ii*), (*iii*) and (*iv*); the values were 234°·91, 233°·73, 233°·78 and 234°·52; the mean value of the periphery was found to be 234°·23.

Again taking 234° to the value of the *sighra* periphery, the corresponding equations for 28°, 60° and 90° of *sighra* anomaly were worked out and the results were as follows:—

Sighra anomaly =	28°	60°	90°
Corresponding *Sighra* Equation =	10°58′	23°1′	33°1′·5

I must say here in justice to the Sanskrit commentator Amraj that he has in almost all cases been able to give from the proper authorities, the appropriate values of the various elements. It seems now clear that the essentials of the lost work of Aryabhata are to be found from the *Khandakhadyaka* of Brahmagupta. So candid is the author that he either states directly or it may be easily inferred, where he differs from Aryabhata.

5. The *Khandakhadyaka* and the *Suryasiddnanta* of Varahamihira.

The striking similarity in the astronomical elements of these two works in certain respects was noticed by Dr. Thibaut in his introduction to the Panchasiddhantika, pp. xix and xx. I have shown above that they agree on all important points. In the light of the present paper we are perhaps to take *cum grano salis* Varaha's following remarks about the five *siddhantas* he summarises:—"The Siddhanta made by Paulisha is accurate; near to it stands the Siddhanta proclaimed by Romaka; more accurate is the Savitra; the remaining ones are far from the truth." We are perhaps to understand by the *Savitra* being *more accurate* that it was made more accurate by Varaha himself by borrowing the astronomical elements from Aryabhata. Any one who looks critically through the second chapter of the modern *Suryasiddhanta* recognises two distinct planetary theories. The first is undoubtedly the older theory; the second the epicyclic theory; and I feel strongly in favour of the hypothesis formulated perhaps for the first *time that the genuine old Suryasiddhanta had nothing of the epicyclic theory in it.* It is an interpolation in the modern book—it was an interpotation by Varahamihira in the older work.

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THE EVECTION AND THE VARIATION OF THE MOON IN HINDU ASTRONOMY

BY

DHIRENDRANATH MUKHOPADHYAYA

(*Daulatpur*)

The ancient Hindus and Greeks were aware of the *Equation of centre* of the moon. Ptolemy (A.D. 140) is credited with having discovered one of the largest and the most important inequality of the moon known as the Evection. This was the only perturbation known to the Greeks. As this perturbation may affect the time of an eclipse by about 6 hours, it was the first to attract the attention of the ancient astronomers. The other large inequality known by the name of the Variation was unknown to the Greek astronomers. As this inequality becomes zero at the full and new moon and therefore does not affect the time of an eclipse, so it was missed by the Greeks. Some writers suppose that it was possibly discovered by an Arabian astronomer, Aboul Wefa in 975 A.D. and lost sight of until Tycho Brahe in A.D. 1580 detected this for the first time in Europe.

Bījopanaya: authorship attributed to Bhāskarācārya.

A short treatise on the corrections of the moon entitled "*Bījopanaya*" has recently been published by the Punjab Sanskrit Book Depot of Lahore with an Introduction by Dr. Ekendranath Ghosh. A careful examination of the corrections given therein has convinced me that these are the deficit of the Equation of centre of the moon as given by Hindu astronomers and the variation of the moon as understood by modern astronomers. The authorship of this treatise has been attributed to the celebrated Hindu astronomer Bhāskarācārya. Ordinarily we know of Bhāskara's *Līlāvatī*, *Bīja-gaṇita* and the *Gaṇita* and *Gola* chapters on astronomy, all comprising his *Siddhānta Śiromaṇi*. As the additional corrections of

the moon are not to be found in the available copies of this work, the publishers should have taken a little more care while placing an entirely new work before the learned public. They should have mentioned from where the manuscript copy was found, how many such copies exist and such other details. In reply to my query to the publishers I was informed that the manuscript copy was supplied to them by a gentleman of Madras and that the copy was very old.

External evidence in support.

I find in Oppert's *Catalogue of Sanskrit MSS. in Southern India* the name of a treatise on astronomy entitled *Bījopanaya* (*Jyotisha*) No.... in the possession of the Saraswati Bhandar at Melkote (Mysore). Aufrecht in his *Catalogus Catalogorum of Sanskrit MSS.* mentions this as one on Algebra evidently mistaking the name for *Bījagaṇita*. The publishers have printed along with the *Bījopanaya* another work *Tithi Nirṇayakārikā* by Śrī Nivāsācārya. From what Śrī Nivāsa writes at the end of the kārikā,[1] we learn that he was born in Śaka 1169=A.D. 1247 and he had the titles of 'Veda Vedānta Deśika' and 'Sarva Tantra Swatantra' from his Guru Veṅkateśa and he was himself a worshipper of god Veṅkateśa. Now we know that this Veṅkateśa or Veṅkatanātha himself possessed the above titles and had written numerous treatises on various subjects. Śrī Nivāsa wrote commentaries on several of Veṅkatanātha's works and also wrote original treatises on various subjects. Now Veṅkatanātha flourished during 1268-1339 A.D. when the Yādava rulers were flourishing in Mysore. Veṅkateśa wrote a kāvya entitled the "*Yādavābhyudaya*" ("The Rise of the Yādavas") evidently in praise of the Yādava rulers who had sometime ago become followers of the great Vaiṣṇava sage Rāmānuja (1117 A.D.). Outwardly, I am informed, the kāvya was written on Kṛṣṇa or Yādava of the Purāṇas, whose descendants the Jādava rulers considered themselves to be. From the nature of books written by Veṇkateśa it is evident that he was a follower of Rāmānuja. From Śrī Nivāsa's 'Śuddhi Dīpikā' we learn that he was the court paṇḍit of Mahītāpana.[2] Now this Mahītāpana is

[1] 'इति कौशिकगोत्रेण श्रीगोविन्दाचार्य्यसूनुना। श्रीवेङ्कटेश पादाब्जसमाराधनकारिणा ॥
तद्दत्तवेदवेदान्तदेशिकाख्येन धीमता। सर्व्वतन्त्रस्वतन्त्रेण भर्गचक्राङ्कजन्मना ॥
श्रीवैखानसनिष्ठेन श्रीनिवासाख्ययज्वना। रचिता परिपूर्णेयं तिथिनिर्णयकारिका ॥'

[2] '...मह्मीतापनीय सभापण्डित...।'—*Śuddhi Dīpikā* by Śrī Nivāsa.

certainly another scholarly way of saying Mahīsūra (Mysore). *Tāpana* is "*Śūra*" or "*Sūrya*," the sun.[1] Śrī Nivāsa was a great paṇḍit as his titles prove. He wrote something in his '*Tithi Nirṇayakārikā*' which evidently refers to his *Suddhi Dīpikā*.[2] We find him mentioning Bhāskara and his *Bījopanaya* in his *Tithi Nirṇayakārikā*.[3] We know Bhāskara lived about A. D. 1150 and was also the court astronomer of the Yādava rulers. From the above considerations it seems natural for Śrī Nivāsa to mention Bhāskara and his "*Bījopanaya*."

Internal evidence in support.

Moreover we have internal evidence in the "*Bījopanaya*" to show the genuineness of the authorship of the book. The 35th verse of this work is a repetition of the first verse[4] of the *Spaṣṭādhikāra* of the *Siddhānta Śiromaṇi*. Here he states clearly the necessity of the coincidence of the computed and observed places of the celestial bodies and remarks that the additional corrections of the moon are more fully treated of in this treatise. The author begins his "*Bījopanaya*" with an introductory note which runs thus: "उक्तं सपरिकरं करणं सोपपत्तिकं गणितगोलयोः। इदमत्र परिशिष्टं वक्तव्यं यत् वीजोपनीतिरहस्य प्रपञ्चनम्। तत् हि मध्याधिकारान्ते अत्र वक्ष्यमाणम् उपजीव्य संचिप्तम्।" "I have already described the treatise on the Gaṇita and Gola with the related subjects (*i.e.*, the *Bījagaṇita* and the *Līlāvatī*) with proofs. Now will be described as an appendix the secrets of finding the positions (of the moon). Desirous of explaining in this treatise I

[1] Vide Amarakośa '…सूर सूर्य्य अर्य्यमादित्य…' and the notes '…सूरेति दन्त्यादिरियम् 'शूरस्तालव्यवान् इति पुरुषोत्तम उणादिवृत्तौ।' 'शूरः सूरोऽपि चादित्य इति द्विरूपकोष:।'

[2] '…अधिकं त्वत्र वक्तव्यं अन्यत्रैव प्रपञ्चितम्।'—Verse 94.

[3] '…वीजोपनयभाष्ये च भास्करेण विपश्चिता।'— Verse 7.

'…वीजोपनयभाष्यान्ते भास्कराद्यैः प्रपञ्चिता। साधितैश्चरवीजान्तैः खार्षेदृक्कर्य्यभिरग्रैः।'—Verse 10.

'तत्कालवीजोपनयो भास्करेण विपश्चिता। वीजोपनयभाष्यादौ अन्यैरपि निवेदितः।'—Verse 24.

[4] 'यात्राविवाहोत्सवजातकादौ खेटैः स्फुटैरेव फलस्फुटत्वम्।
स्यात् प्रोच्यते तेन नभश्चराणां स्फुटक्रियादृग्गणितैक्यकृद् या॥

अथेदानीं स्फुटाधिकारोक्त स्पष्टीकरणस्य चरवीजसंस्कारपर्य्यन्ततां ज्ञापयितुं तदधिकारोपक्रमोक्तं श्लोकेनेमं प्रकरणं संगमयन्नाह यात्रेति। प्रोच्यते सर्व्वैः शास्त्रैरिति प्राकरणिकोऽर्थः। या दृग्गणितैक्यकृत् सा—स्फुटक्रिया इति योजना। एतेन स्पष्टाधिकारोक्तस्फुटक्रिया न तत्र परिसमाप्ता किन्तु अत्रैव परिसमाप्ता इति ज्ञापितम्।'

cut that matter short at the end of the chapter on *Madhyamādhikāra* (of the *Grahagaṇita*).''[1] In verse 7 of the "*Bījopanaya*" the author states the year of his birth as *Śaka* 1036 (A.D. 1114) and says that he finished this treatise when he completed his 37th year.[2] From Bhāskara's "*Golādhyāya*" we learn that he finished his "*Sidddhānta Śiromaṇi* when he attained his 36th year.[3] From this it is evident that Bhāskara continued his observations for two years more and then wrote the present treatise. Verses 43 to 50 of the *Bījopanaya* are repetitions from his "*Golādhāya*" chapter on Eclipses (verses 11-16) and these are also found in his '*Grahagaṇita*' chapter on solar Eclipses.

In these he fully explains that parallax correction is unnecessary in 'tithis' which begin or end at the same absolute instant of time all the earth over.[4] The language and style of the present treatise also bears a striking resemblance to that of the "*Siddhānta Śiromaṇi.*"

Objections met.

Now a question may be raised why are not these additional corrections of the moon to be found in Bhāskara's *Karaṇakutuhala*, a practical treatise, the epoch of which is Śaka 1105 (=1183 A.D.), that is, when Bhāskara was 69 years of age. We must remember that this short practical treatise was meant to satisfy the curiosity of the beginners as its very name *Karaṇakutuhala* indicates. Hence it is no wonder that these intricate corrections do not find a place there.

Bhāskara's Corrections explained.

Now I shall explain the additional corrections of the moon as found in the "*Bījopanaya.*" In verse 8 the author speaks of the maximum additional correction as $\pm 112'$.[5] Later on Bhāskara in

[1] Compare what he says at the end of the "*Madhyamādhikāra.*"
'...संक्षिप्तं न च विस्तृतं विरचितं...।'—Verses 9-10, '*Grahagaṇita.*'

[2] 'रसगुणाभ्रमहीशकवत्सरे ममजन्म वभूव महीतले। नगगुणोन्मितवत्सरपूरणे यदुत वीजफलं तु मयोन्मितम्।'

[3] 'रसगुणपूर्णमहीसमशकसमयेऽभवन् ममोत्पत्तिः। रसगुणवर्षेण मया सिद्धान्तशिरोमणी रचितः।'
—Verse 58, '*Praśnādhyāya.*'

[4] '...युगपत्प्रवर्त्तमानानां तिथ्यादीनां न लम्बनसंस्कारप्रसक्तिः।'

[5] 'लिप्ताविधोरर्कमहीमिता मे दृग्गोचराः प्रत्यहमीक्षितस्य।
कदम्बगोलागतसूत्रपाते क्रान्तौ धनर्णत्वजुषो भमध्यात्॥'

his own lucid and forceful style resolved the correction into its two components and showed how they vary and in what positions they are maximum or zero, just as he had done in explaining the Equation of time by resolving it into its two components, long before Flamsteed, the first Royal Astronomer of England (1700 A.D.) who is credited by European astronomers to have explained this for the first time. Bhāskara's own notes on verses 20-25 of the *Bījopanaya* are very interesting reading. He states '...तत्र प्रथमफलस्य प्रवृत्तिनिवृत्तिश्च तुङ्गादेव मन्दफलसमानयोगक्षेमा।' "Of these the beginning and end of the first equation are similar (rise or fall—positive or negative) to the equation of the centre (of the moon) beginning from the apogee (of the moon). This is the supplement of the equation of centre of the moon as given by the Hindu astronomers. This is stated by them as 5°5′ (about). The modern correct value of this is 6°17′. So that the deficit of the Hindu value is 1°12′ (about). Bhāskara gives this in his first correction and states this to be 1°18′, 6′ more than the true value but we should remember that for the second correction he gives 6′ less than the modern value, thus the combined maximum correction found by him is correct.

The second correction which is clearly the variation is neatly explained by Bhāskara thus : अनन्तरस्य तु स्फुटसूर्यस्थानात् ओजेऽधनं युग्मे ऋणं इति प्रवृत्तिनिवृत्तिश्च उपलभ्येते।' "The beginning and end of the other equation are observed as positive (with the moon) in the odd quadrants and negative in the even quadrants beginning from the position of the apparent sun." This is exactly what modern astronomers say 'The variation sets the moon ahead between new moon and first quarter and between full moon and last quarter, and behind in the other two quarters of the month.' The combined maximum effect of the two corrections according to modern astronomy is (72′+40′) or 112′, exactly the same as found by Bhāskara.

Now I shall quote here verses 20—25 and translate the same:

'तुङ्गादाद्यपदान्तस्थाद् विधोरर्के पदार्धतः। परमं चन्द्र वैषम्यं ऋणत्वेन समीक्ष्यते ॥२०॥

तत्तृतीयपदान्तस्थात् पृष्ठगेऽर्के पदार्धतः। परमं चन्द्र वैषम्यं धनत्वेन समीक्ष्यते ॥२१॥

चन्द्रतुङ्गे च नीचे च शशाङ्कार्कग्रहौ यदि। मन्दस्फुटगतश्चन्द्रो निर्बीजस्तुल्यमीक्ष्यते ॥२२॥

ओजान्तयोर्विधोस्तुङ्गाच्छशाङ्कार्कग्रहौ यदि। चतुस्त्रिंशत्कलाहीनं वैषम्यं तु समीक्ष्यते ॥२३॥

अग्रतः पृष्ठतोवापि रवेश्चन्द्रे पदार्धगे। तुङ्गतुल्ये चतुस्त्रिंशत् कलावैषम्यमीक्ष्यते ॥२४॥

एवं तन्नीचतुल्येऽपि वैषम्यं तावदेव हि। एवं व्यासात् समासाच्च पौनःपुन्येन वेधनात् ॥

चरबीजमिदं क्लृप्तं मया सद्भिः समीक्ष्यताम् ॥२५॥[2]

20. "With the moon placed at the end of the first quadrant after its apogee and the sun placed half a quadrant in front of the moon, the maximum subtractive difference ($-112'$, as stated in verse 8) is observed in the position of the moon.

21. "With the moon placed at the end of the third quadrant after its apogee and the sun placed half a quadrant behind the moon, the maximum additive difference ($+112'$) is observed in the position of the moon.

22. "If the sun and the moon (jointly or separately) are placed at the apogee and the perigee of the moon, then the moon corrected by the equation of centre has no other corrections to make, *i.e.*, the total correction is zero.

23. "If the sun and the moon either jointly or separately are placed at the end of odd quadrants after the moon's apogee, a difference of $34'$ less (than $112'$, *i.e.*, $\pm 78'$) is observed in the position of the moon.

24 and 25. "With the sun placed at half a quadrant in front of or behind the moon which coincides with its apogee or the perigee, a difference of $\pm 34'$ is observed in the position of the moon. Thus by repeated observations and by combination and resolution into the components, I find these 'Cara Vījas.' Let the learned observe and verify these."

Now Professor Brown's modern formula for the true longitude of the moon is

$$\lambda = L + 377' \sin l + 13' \sin 2l + 76' \sin (2D - l) + 40' \sin 2D - 11' \sin l' + \ldots\ldots\ldots$$

where L is the moon's mean longitude ; l, the distance of the mean moon from the mean perigee ; D, its distance from the mean sun ; and l', the distance of the latter from its perigee. The terms $377' \sin l + 13' \sin 2l$ give us the equation of centre.

$76' \sin (2D - l)$ is the evection and $40' \sin 2D$ is the variation.

I. For verse 20 we have

$$72' \sin 270° + 40' \sin (2 \times 315°) = -72' - 40' = -112'.$$

The correction of Evection in this position is

$$76' \sin (2 \times 315° - 270°) = 0'.$$

Hence the observed moon's position as stated by Bhāskara is true.

II. For verse 21 we have

$$72' \sin 90° + 40' \sin (2 \times 45°) = 72' + 40' = +112'.$$

The correction of evection in this position is

$$76' \sin (2 \times 45^\circ - 90^\circ) = 0'.$$

Hence the observed moon's position as stated by Bhāskara is true.

III. From verse 22 we have

$$72' \sin (0^\circ \text{ or } 180^\circ) + 40' \sin 2\,(180^\circ \text{ or } 0^\circ) = 0'.$$

The correction of evection in these positions is

$$76' \sin (2 \times 180^\circ \text{ or } 2 \times 0^\circ - 0^\circ \text{ or } 180^\circ) = 0'.$$

Hence the position of the moon as observed by Bhāskara is true.

IV. From verse 23 we have

$$72' \sin (90^\circ \text{ or } 270^\circ) + 40' \sin (2 \times 0^\circ \text{ or } 2 \times 180^\circ) = \pm 72'.$$

Here Bhāskara's value is $\pm 78'$. But for the observed position of the moon the correction of evection must also be taken account of. For the above positions the evection is

$$76' \sin (2 \times 0^\circ \text{ or } 2 \times 180^\circ - 90^\circ \text{ or } 270^\circ) = \mp 76'.$$

So that the difference between the observed and calculated places in these positions should be $\pm$ 4′ or practically nil. Bhāskara has evidently erred here. He could not discover the Evection. In these positions the moon is either full or new. During the new moon no observations can be made unless there be an eclipse of the sun. The correction of refraction was unknown to the Hindu astronomers. Hence any observation with the sun and the moon on the horizon on the full moon day must be vitiated by double the amount of the horizontal refraction (about 70′). Bhāskara seems to have combined the two corrections, the supplement of the equation of centre of the Hindu value and the variation in these positions and stated the combined effect as the difference to be observed.

V. From verses 24 and 25 we have

$$72' \sin (0^\circ \text{ or } 180^\circ) + 40' \sin 2 \times (\pm 45^\circ) = \pm 40'.$$

Here again the evection in these positions is

$$76' \sin \{2 \times (\pm 45^\circ) - 0^\circ \text{ or } 180^\circ\} = \pm 76'.$$

Therefore the total difference between the mean and the true places would be $\pm 36'$ or $\pm 116'$. Bhāskara's value is $\pm 34'$. Here also as in the former case Bhāskara seems to have found out the correct position only partially. If Bhāskara had directed his attention to the records of solar and lunar eclipses only, for several years

he might easily have discovered the combined effect of the supplement of the equation of centre (of the Hindu value) and the evection. The former one being already known to Bhāskara he could easily have found the evection. It seems Mañjula directed his attention to these and thus stated the combined effect of the supplement to the equation of centre and the evection as will be evident soon.

Now Bhāskara's note when the first correction, *i.e.*, the supplement to the equation of cente, is maximum or zero is apparent. We know that this is maximum when l is $\pm 90°$ and Bhāskara has given this correction at intervals of 3°45′ up to 90° in verses 26-28, which according to him are 6, 13, 21, 27, 33, 39, 45, 51, 56, 61, 65, 68, 70, 72, 74, 75, 75, 76, 76, 77, 77, 78, 78, 78 minutes respectively. (Here we should remember that for intermediate positions $13' \sin 2l$ has got to be taken account of in addition to $72' \sin l$.) Bhāskara directs the above values found by him to add or subtract from the equation of centre (5° 5′) according as they are additive or subractive.

We know that the variation is maximum when $2D = \pm 90°$ and Bhāskara has given this correction at intervals of 3°45′ up to 90° in verses 29 to 32, which he found to be 6, 9, 13, 17, 22, 24, 27, 30, 32, 33, 34, 34, 34, 33, 31, 28, 26, 24, 20, 16. 11, 8, 3, 0 minutes respectively showing clearly that the maximum occurs at the octants. These are additive between new moon and first quarter and between full moon and last quarter and subtractive in the other two quarters of the month. Bhāskara subtracts here the longitude of the moon from that of the sun. Thus the 'tithi' being negative he writes here correctly 'subtractive in the odd quadrants and additive in the even quadrants.' Bhāskara finds the variation to be 34′. But Herschel in 1873 A.D. stated this to be not less than 32′ and Baily in his astronomical Tables and Formulæ put this at 35′. The latest value as found by Professor Brown is 40′.

Other important matters in the Bījopanaya.

The treatise is illuminating throughout and much may be learnt on a perusal of the text and its commentary. For instance the definition of a 'tithi' is very vague as given in the *siddhāntas.* They simply state that it is to be determined from the true positions of the moon and the sun. The position of the moon may be the point on the ecliptic where the declination circle or the circle of

celestial latitude through the moon cuts the ecliptic. These points are nowhere coincident except when the moon is on the solstitial colure or at one of its nodes. Then there may yet be another possibility. The 'tithi' may be the angular distance between the moon and the sun or it may as well be the difference of the true longitudes of the two; the moon's longitude being reckoned from the vernal equinox to the node on the ecliptic and thence from the node to the actual position of the moon along its own orbit, which some astronomers would like to take. The mean longitudes in the Nautical Almanacs are stated in this manner. But Herschel deprecates this mode of reckoning 'What is most improperly called in some books the longitude of perihelion on the orbit is a broken arc or an angle made up of two in different planes, *viz.*, from the equinox to node on the ecliptic and thence to the perihelion on the orbit.'[1] A perusal of Bhāskara's *Bījopanaya* clears out all doubts and we understand the 'tithi' to be the difference of true longitudes of the moon and the sun on the ecliptic throughout.

Earlier History : Mañjula.

We know Bhāskara mentions Mañjula (commonly styled Muñjala) and other ancient astronomers several times in his *Siddhānta Śiramoṇi*. The late Sudhākara Dvivedī, astronomer of Benares, mentioned in his *Gaṇaka Taraṅgiṇī* (Lives of the Hindu astronomers) that Mañjula (A.D. 932) gives a new correction in addition to the moon's equation of centre, to find its true place. Professor Joges Chandra Roy, late of the Ravenshaw College, Cuttack, in his 'আমাদের জ্যোতিষী ও জ্যোতিষ', remarks on this that it was unexplainable why then Bhāskara was silent on this correction. The publication of the *Bījopanaya* shows that Bhāskara wrote a separate treatise on the additional corrections of the moon. Mañjula's Laghumānasa has not yet been published. However, I am indebted to Dr. Bibhutibhushan Datta of the Calcutta University who very kindly supplied me with the verses dealing with the corrections of the moon from the copy of the manuscript of Mañjula's *Laghumānasa* at his disposal. I shall explain presently what these corrections are. We also learn from Dvivedī's *Gaṇaka Taraṅgiṇī* that Nityānanda in 1693 A.D. mentions an additional correction of the

[1] Herschel, *Outlines of Astronomy*, p. 329 footnote.

moon styled the *Pākṣika Saṁskāra* (fortnightly corrections). I also learn from a Paṇḍit that an additional correction of the moon is also to be found in the *Siddhānta Samrāṭa* of Paṇḍit Jagannātha, the court astronomer of Rājā Jai Singh of Jaipur written in A.D. 1729. This treatise (perhaps wrongly written as *Samrāṭa Siddhānta* was 'undertaken' for publication by the Gaekwar's Oriental Library, Baroda along with the *Siddhānta-sāra-Kauśtubha* by the same author which was advertised as being 'in the press' long ago. But as none of these have appeared, nothing could be said now.

Now I come to the explanation of the corrections of the moon as given by Mañjula.

भुजो लिप्तीकृतच्छेदभक्तो ग्रहफलांशका: ।......॥७॥

"The Bhujajyā (*i.e.*, $R°$ sine of the distance of the planet from the apogee, *i.e.*, the anomaly according to Hindu astronomers) is turned into minutes and this is divided by the 'cheda' (a number which is different for different planets). The result gives us the equation of centre in degrees." In the case of the moon the 'cheda' is given by the commentator Praśastidhara as 97 and from values of sin θ given there it is evident that he has taken for R the value 491 which is exactly one seventh of 3438, the usual value. Therefore $R°$ sin $90° = 8°\ 11'$ or $491'$. Thus the maximum value of the equation of centre of the moon comes out to be $\frac{491'}{97}$ sin $90° =$ $5°4'$ (about). This is almost the same as in other Hindu astronomical treatises.

Now in verses 11 and 12 Mañjula gives another correction for the moon (also for its daily motion).

इन्दूच्चोनार्ककोटिघ्ना-गत्यंशा-विभवा-विधो: ।
गुणोव्यर्केन्दु दो: कोट्यो-रूपपञ्चाप्तयो: क्रमात् ॥११॥
फले शशाङ्क तद्गत्यो-र्लिप्तादे स्वर्णयो द्वंघे ।
ऋणं चन्द्रे धनं भुक्तौ स्वर्णसाम्यवधेऽन्यथा ॥१२॥

"Multiply the cosine of the distance of the sun from the apogee of the moon by the mean daily motion of the moon minus 11. Call this the multiplier. The sine and the cosine of the distance of the moon from the sun is divided respectively by 1 and 5. These two results are multiplied respectively by the multiplier found above.

The results are the corrections of the moon and of its mean daily motion respectively in minutes. If the multiplier and the multiplicand are one positive and the other negative the correction for the moon is subtractive and that of the daily motion is additive. While if both are positive or negative the corrections are otherwise, *i.e.*, additive for the moon and subtractive for the daily motion." Therefore Mañjula's formulas for the corrections become :—

$$R^{\circ} \cos \{-(D-l)\} \times (13{\cdot}166-11) \times \frac{R}{1} \sin D \ldots (\text{A}) \quad (\text{for the moon}),$$

$$\text{and } R^{\circ} \cos \{-(D-l)\} \times (13{\cdot}166-11) \times \frac{R}{5} \cos D \ldots (\text{B})$$

(for its daily motion),

where l=the distance of the mean moon from its mean perigee and D=the distance of the mean moon from the mean sun. The mean daily motion of the moon$=13^{\circ}{\cdot}166\ldots$ Therefore formula (A) becomes $8{\cdot}18\ldots \times \cos (D-l) \times 2{\cdot}166 \times 8{\cdot}18..\ \sin D$.

$$=8{\cdot}18 \times 8{\cdot}18 \times 2{\cdot}166 \times \cos (D-l) \sin D.$$

$$=67 \times 1{\cdot}083 \times 2 \cos (D-l) \sin D.$$

$$=72{\cdot}5 \times 2 \cos (D-l) \sin D.$$

$$=72{\cdot}5\{\sin (2D-l) + \sin l\}$$

$$=72'{\cdot}5 \sin (2D-l) + 72'{\cdot}5 \sin l.$$

Now we know from modern astronomy that $76' \sin (2D-l)$ is the evection and $377' \sin l$ is the equation of centre of the moon. But the value of this equation of centre is $5^{\circ}4'$ ($304'$) as found before. So that the deficit of the Hindu value is ($377'-304'$) or $73'$ from modern figures. It is now perfectly clear that this second correction of Mañjula is the combined effect of the evection and the deficit of the equation of centre only with slight variations in the constants. This formula giving the joint effects of the above corrections shows Mañjula's great ingenuity.

Bhāskara's credit.

Evidently Mañjula could not discover the other inequality known as the Variation. It was left to Bhāskara to discover this and to explain this very clearly in a manner befitting that celebrated astronomer.

In conclusion I acknowledge my indebtedness to Prof. Asutosh Mitra, M.A., late of the Vidyasagar College, Calcutta and to Dr. Bibhutibhushan Datta, D.Sc., P.R.S. of the Calcutta University for valuable help in completing this paper.

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7

On Some Integral Inequalities.

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(*Read 10th August, 1930.*)

1 The object of the present paper is to establish a class of inequalities between the *quotients* of integrals of two or more positive continuous functions and the integrals of the quotients of the functions taken in pairs. The corresponding problem for the *product* of the integrals of two functions and the integral of their product was first stated by Tchebycheff * in the form

$$(1) \qquad \int_0^1 f(x)g(x)dx > \int_0^1 f(x)dx \int_0^1 g(x)dx$$

where $f(x)$ and $g(x)$ are positive continuous functions of x such that they both increase continuously or both decrease continuously. But if one of the functions be increasing and the other decreasing in the given interval, the sign of the inequality is reversed. This theorem holds good also in the case of any real and finite limits of the variable x. Thus we have

$$(2) \qquad \int_{x_1}^{x_2} f(x)dx \int_{x_1}^{x_2} g(x)dx < \text{ or } > (x_2-x_1)\int_{x_1}^{x_2} f(x)g(x)dx$$

under the same set of conditions. Dunkel † has given generalisations of the form (2) to cover any number of functions, and has applied his results to the problems of minimizing certain definite integrals.

* See Hermite—*Cours professé pendant le 2' Semestre*, 1881-82. A proof of Tchebycheff's inequality by Picard is also given here.

† O. Dunkel.—"Integral Inequalities with applications to the calculus of variations." *The American Mathematical Monthly*, 31 (1924), pp. 326—337.

In the present paper, I have firstly dealt with two positive continuous functions $f(x)$ and $g(x)$ and have established the inequality

$$\int_{x_1}^{x_2} \frac{f(x)}{g(x)} dx \leq \text{ or } \geq (x_2 - x_1) \frac{\int_{x_1}^{x_2} f(x) dx}{\int_{x_1}^{x_2} g(x) dx}$$

holding under conditions similar to these of Tchebycheff. Illustrations have been inserted from analytic, circular, and transcendental functions. Secondly the result has been extended to include any number of functions and any powers of functions, and lastly several known results have been deduced with the help of the theorems proved by me.

I take this opportunity to express my best thanks to Dr. Ganesh Prasad for his kind interest and constant encouragement.

§ 1.

Fundamental Theorem.

2. THEOREM I. *If $f(x)$ and $g(x)$ be two positive continuous functions of x, defined in the interval $x_1 \leq x \leq x_2$, then will,*

$$(1.1) \qquad \int_{x_1}^{x_2} \frac{f(x)}{g(x)} dx \leq \text{ or } \geq (x_2 - x_1) \frac{\int_{x_1}^{x_2} f(x) dx}{\int_{x_1}^{x_2} g(x) dx}$$

according as $\frac{f(x)}{g(x)}$ and $g(x)$ both increase (or both decrease), or one increases and the other decreases, in the given interval.

Before proceeding with the proof of this theorem, we shall try to find out the corresponding theorem in finite series.

Suppose a_ν and b_ν denote positive numbers for $\nu=1, 2, 3, \ldots n$. It can be easily seen that

$$\sum_{\nu=1}^{n} \frac{a_\nu}{b_\nu} \sum_{\nu=1}^{n} b_\nu = n \sum_{\nu=1}^{n} a_\nu + \tfrac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \left(\frac{a_i}{b_i} - \frac{a_j}{b_j} \right) (b_j - b_i)$$

Now if $\frac{a_\nu}{b_\nu}$ and b_ν both increase, then the second summation on the right-hand side is negative; so also if they both decrease. But if one of the two increases while the other decreases, the summation is positive. From this we find that

$$(1.2) \qquad \sum_{\nu=1}^{n} \frac{a_\nu}{b_\nu} \sum_{\nu=1}^{n} b_\nu \leq \text{ or } \geq n \sum_{\nu=1}^{n} a_\nu$$

according as $\frac{a_\nu}{b_\nu}$ and b_ν both increase (or both decrease), or one of them increases and the other decreases.

Hence we have

$$(1.3) \qquad \sum_{\nu=1}^{n} \frac{a_\nu}{b_\nu} \leq \text{ or } \geq n \frac{\sum_{\nu=1}^{n} a_\nu}{\sum_{\nu=1}^{n} b_\nu}$$

under conditions set forth above. Equality holds when $\frac{a_\nu}{b_\nu}$ is a constant or simply b_ν is a constant.

Now the transition from (1.3) to the case of integrals can be easily effected by considering the numbers in the series to be infinite.

Put $a_\nu = f_\nu(x_1+\nu h)$,

and $b_\nu = g_\nu(x_1+\nu h)$, $\qquad \nu=1,2,3,\ldots\ldots n$

where $h = \frac{x_2-x_1}{n}$

From (1.3) we have

$$\Sigma\, h \frac{a_\nu}{b_\nu} \leq \text{ or } \geq nh \frac{\Sigma\, h a_\nu}{\Sigma\, h b_\nu}.$$

$$(1.1) \quad i.e. \quad \int_{x_1}^{x_2} \frac{f(x)}{g(x)}\, dx \leq \text{ or } \geq (x_2-x_1) \frac{\int_{x_1}^{x_2} f(x)dx}{\int_{x_1}^{x_2} g(x)dx}.$$

This result is also directly deduced as follows :—

Consider the integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left[\frac{f(x)}{g(x)}-\frac{f(y)}{g(y)}\right]\left[g(x)-g(y)\right]dx\,dy. \qquad \begin{cases} x_1=y_1 \\ x_2=y_2 \end{cases}$$

This can be seen to be equal to

$$2(x_2-x_1)\int_{x_1}^{x_2}f(x)dx - 2\int_{x_1}^{x_2}\frac{f(x)}{g(x)}dx \int_{x_1}^{x_2}g(x)dx.$$

Hence
$$\frac{\displaystyle\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left[\frac{f(x)}{g(x)}-\frac{f(y)}{g(y)}\right]\left[g(x)-g(y)\right]dx\,dy}{\displaystyle\int_{x_1}^{x_2}g(x)\,dx}$$

$$(1.4) \qquad = 2(x_2-x_1)\frac{\displaystyle\int_{x_1}^{x_2}f(x)dx}{\displaystyle\int_{x_1}^{x_2}g(x)dx} - 2\int_{x_1}^{x_2}\frac{f(x)}{g(x)}dx.$$

Now if both $\frac{f(x)}{g(x)}$ and $g(x)$ increase or both decrease, we have the left hand side of (1.4) positive, whereas it is negative if one of these increases while the other decreases. Hence we have

$$(1.1) \qquad \int_{x_1}^{x_2}\frac{f(x)}{g(x)}dx \leq \text{ or } \geq (x_2-x_1)\frac{\displaystyle\int_{x_1}^{x_2}f(x)\,dx}{\displaystyle\int_{x_1}^{x_2}g(x)\,dx}$$

according as $\frac{f(x)}{g(x)}$ and $g(x)$ both increase (or both decrease), or one increases while the other decreases.

From (1.4) it is also evident that equality occurs only when $\frac{f(x)}{g(x)}$ or $g(x)$ is a constant.

3. *Note* 1. Putting $f(x)$ in place of $\dfrac{f(x)}{g(x)}$, we have

$$(1.5)\quad \int_{x_1}^{x_2} f(x)dx \leq \text{ or } \geq (x_2-x_1)\ \frac{\displaystyle\int_{x_1}^{x_2} f(x)g(x)dx}{\displaystyle\int_{x_1}^{x_2} g(x)dx},$$

$$i.e.,\quad \int_{x_1}^{x_2} f(x)dx \int_{x_1}^{x_2} g(x)dx \leq \text{ or } \geq (x_2-x_1) \int_{x_1}^{x_2} f(x)g(x)dx$$

which is the inequality of Tchebycheff.

Equality occurs when $f(x)$ or $g(x)$ is a constant.

Note 2. Again putting $f(x)$ equal to some constant k in (1.1), and taking $g(x)$ as an increasing or decreasing function of x, we have,

$$\int_{x_1}^{x_2} \frac{k}{g(x)}dx \geq (x_2-x_1)\ \frac{\displaystyle\int_{x_1}^{x_2} kdx}{\displaystyle\int_{x_1}^{x_2} g(x)dx},$$

which can be put as

$$(1.6)\quad \int_{x_1}^{x_2} g(x)dx \int_{x_1}^{x_2} \frac{dx}{g(x)} \geq (x_2-x_1)^2$$

Equality occurs when $g(x)$ is a constant.

Note 3. Expressed in terms of Mean values, the theorem states—

The mean value of the quotient of two positive continuous functions is less than or greater than the quotient of their mean values, according as the quotient of the functions and the function in the denominator both increase (or decrease), or the one increases while the other decreases.

Illustrations and Applications.

4. *Illustrations.*

(1) Suppose $f(x)=x^n$ and $g(x)=x^m$ where n and m are positive numbers. Then

$$\int_0^x x^{n-m}\,dx < \text{ or } > x\,\frac{\int_0^x x^n dx}{\int_0^x x^m dx}$$

according as $m <$ or $> n$.

Equality occurs in this case when either $n=m$ or $m=0$.

(2)
$$\int_0^{\frac{\pi}{2}} \tan x\,dx > \frac{\pi}{2}\,\frac{\int_0^{\frac{\pi}{2}} \sin x\,dx}{\int_0^{\frac{\pi}{2}} \cos x\,dx}$$

$$\int_0^{\frac{\pi}{2}} \cot x\,dx > \frac{\pi}{2}\,\frac{\int_0^{\frac{\pi}{2}} \cos x\,dx}{\int_0^{\frac{\pi}{2}} \sin x\,dx}$$

$$\log\sqrt{2} > \frac{\pi}{4}(\sqrt{2}-1).$$

(3)
$$\int_0^x \frac{dx}{e^x+e^{-x}} > x\,\frac{\int_0^x dx}{\int_0^x (e^x+e^{-x})\,dx},$$

$$2\tan^{-1} e^x > \int_0^x \operatorname{sech} x\,dx > \frac{x^2}{\sinh x}. \qquad (x>0).$$

5. *Applications to Gamma Functions and Beta Functions.*

$$\int_0^1 x^{p-1}\,dx > \frac{\int_0^1 x^{p-1}(1-x)^{q-1}\,dx}{\int_0^1 (1-x)^{q-1}\,dx}$$

$$\frac{1}{pq} > \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}.$$

$$\Gamma(p+1)\,\Gamma(q+1) < \Gamma(p+q).$$

$$B(p,q) < \frac{1}{pq}.$$

$$\int_0^\infty \frac{t^{x-1}\,dt}{(1+t)^{x+y}} < \frac{1}{xy}.$$

6. *Inequalities giving upper and lower bounds of $\frac{\pi}{2}$.*

Consider the integral $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$.

Now $$\int_0^{\frac{\pi}{2}} \sin^2 x\,dx < \int_0^{\frac{\pi}{2}} \sin x\,dx < \frac{\pi}{2}\,\frac{\int_0^{\frac{\pi}{2}} \sin^n x\,dx}{\int_0^{\frac{\pi}{2}} \sin^{n-1} x\,dx}$$

by (1.1)

and this again is less than

$$\frac{\pi}{2}\,\frac{\int_0^{\frac{\pi}{2}} \sin^{n-1} x\,dx}{\int_0^{\frac{\pi}{2}} \sin^n x\,dx}$$

for $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$ is a decreasing function for all positive integral values of n.

Take n to be even integer. Then from the above we have

$$(1.61) \qquad \frac{\pi}{4} < \frac{\pi}{2} \cdot \frac{\pi}{2} \, \frac{(n-1)(n-1)(n-3)(n-3)\ldots\ldots 3.3.1}{n(n-2)(n-2)(n-4)\ldots\ldots 4.2.2.}$$

$$< \frac{n(n-2)(n-2)(n-4)\ldots\ldots 4.2.2.}{(n-1)(n-1)(n-3)(n-3)\ldots\ldots 3.3.1.}$$

i.e.,

$$(1.62) \qquad \frac{n(n-2)(n-2)(n-4)\ldots\ldots 4.2.}{(n-1)(n-1)(n-3)(n-3)\ldots\ldots 3.3.} < \frac{\pi}{2}$$

$$< \frac{n(n-2)(n-2)(n-4)\ldots\ldots 4.2.2.}{(n-1)(n-1)(n-3)(n-3)\ldots\ldots 3.3.1.}$$

Again it can be easily seen that

$$\frac{(n-2)(n-2)(n-4)\ldots\ldots 4.2.2.}{(n-1)(n-3)(n-3)\ldots\ldots 3.3.1.} < \frac{\pi}{2},$$

whence we have the stronger inequality *

$$(1.63) \qquad \frac{(n-2)(n-2)(n-4)\ldots\ldots 4.2.2.}{(n-1)(n-3)(n-3)\ldots\ldots 3.3.1.} < \frac{\pi}{2} < \frac{n(n-2)(n-2)\ldots 4.2.2.}{(n-1)(n-1)(n-3)\ldots 3.3.1.}$$

7. *Inequalities in series.*

Take the case of the integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x \, dx}{(a+\sin x)(b+\sin x)}$

where $b>a>1$.

With the help of Theorem I, it can be easily seen that

$$(1.71) \qquad \int_0^{\frac{\pi}{2}} \frac{\cos x \, dx}{(a+\sin x)(b+\sin x)} > \frac{\pi}{2} \cdot \frac{\displaystyle\int_0^{\frac{\pi}{2}} \cos x \, dx}{\displaystyle\int_0^{\frac{\pi}{2}} (a+\sin x)(b+\cos x)\, dx}$$

* If n be made to increase indefinitely in (1.63) we have the upper and lower bounds of $\frac{\pi}{2}$ equal. Thus the value of $\frac{\pi}{2}$ is given by means of an infinite product. This value of $\frac{\pi}{2}$ was first investigated by Wallis in his *Arithmetica Infinitorum*, 1656.

Hence we get

$$\frac{1}{b-a}\log\left\{\frac{b}{a}\cdot\frac{a+1}{b+1}\right\} > \frac{2\pi}{(2ab+1)\pi+4(a+b)}$$

i.e.,

$$(1.72)\qquad \frac{1}{a(b+1)}-\frac{1}{2}\frac{(b-a)}{a^2(b+1)^2}+\frac{1}{3}\frac{(b-a)^2}{a^3(b+1)^3}-\ldots$$

$$> \frac{2\pi}{(2ab+1)\pi+4(a+b)}.$$

In particular,

$$\frac{1}{3}-\frac{1}{2.3^2}+\frac{1}{3.3^3}-\frac{1}{4.3^4}+\ldots \quad > \frac{2\pi}{5\pi+12}.$$

8. *Applications to Incomplete Gamma Function and Confluent Hypergeometric Function.*

Next take the case of Incomplete Gamma Function

$$\gamma(n,x)=\int_0^x t^{n-1}e^{-t}\,dt.$$

We have

$$(1.81)\qquad \int_0^x t^{n-1}e^{-t}\,dt > \text{or} < x.\frac{\int_0^x t^{n-1}\,dt}{\int_0^x e^t\,dt}, \text{ by } (1.1)$$

$$i.e., \qquad > \text{or} < \frac{x^{n+1}}{n(e^x-1)}$$

$$\text{according as} \quad n < \text{or} > 1+\frac{1}{\log\left(1+\frac{1}{t}\right)}$$

Hence

$$\Gamma(n)-x^{\frac{1}{2}(n-1)}\,e^{-\frac{1}{2}x}\,W_{\frac{1}{2}(n-1),\,\frac{1}{2}n}(x) \;> \text{or} < \frac{x^{n+1}}{n(e^x-1)}$$

$$\text{according as} \quad n < \text{or} > 1+\frac{1}{\log\left(1+\frac{1}{t}\right)}$$

where $W_{k,m}(x)$ is a solution of the differential equation *

$$\frac{d^2 W}{dx^2}+\left\{-\frac{1}{4}+\frac{k}{x}+\frac{\frac{1}{4}-m^2}{x^2}\right\} W=0$$

i.e., the function $W_{k,m}(x)$ is defined by the integral

$$-\frac{1}{2\pi i}\Gamma\left(k+\tfrac{1}{2}-m\right)e^{-\frac{1}{2}x}x^k\int_{\infty}^{(0+)}(-t)^{-k-\frac{1}{2}+m}\left(1+\frac{t}{x}\right)^{k-\frac{1}{2}+m}e^{-t}\,dt,$$

or, under certain circumstances by the real integral

$$\frac{1}{\Gamma\left(\frac{1}{2}-k+m\right)}x^k e^{-\frac{1}{2}x}\int_0^{\infty}t^{-k-\frac{1}{2}+m}\left(1+\frac{t}{x}\right)^{k-\frac{1}{2}+m}e^{-t}\,dt.$$

9. *Applications to Elliptic and Legendre Functions.*

Consider the Integral $\int_z^1 \frac{dz}{\sqrt{1-z^2}}$ where $0<z<1$.

By (1.1) we easily get

$$(1.91)\qquad \int_z^1\frac{dz}{\sqrt{1-z^2}}<(1-z)\frac{\int_z^1\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}}{\int_z^1\frac{dz}{\sqrt{1-k^2z^2}}}\quad 0<k<1.$$

Hence we have

$$(1.92)\qquad \left(\frac{\pi}{2}-\sin^{-1}z\right)\left(\sin^{-1}k-\sin^{-1}kz\right)<k(1-z)\left(\operatorname{sn}^{-1}1-\operatorname{sn}^{-1}z\right)$$

$$\text{i.e.,}\quad <k(1-z)(K-\operatorname{sn}^{-1}z).$$

In particular

$$\sin^{-1}z\,\frac{\sin^{-1}kz}{kz}<\int_0^z\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}\qquad \text{i.e.,}\ \operatorname{sn}^{-1}z.$$

* This equation was given by Whittaker, *Bulletin of the American Mathematical Society*, X (1904), pp. 125-134.

Again if we put in (1.92)

$$z=\left(\frac{y-1}{y+1}\right)^{\frac{1}{2}},\ k^2=\frac{(x-1)(y+1)}{(x+1)(y-1)} \text{ where } y>x>1$$

we have

$$(1.93)\quad \frac{1}{1-z}\left(\frac{\pi}{2}-\sin^{-1}z\right)(\sin^{-1}k-\sin^{-1}kz)$$

$$<k\{(x+1)(y-1)\}^{\frac{1}{2}}\sum_{n=0}^{\infty}P_n(x)Q_n(y)$$

$$i.e.,\ <\{(x-1)(y+1)\}^{\frac{1}{2}}\sum_{n=0}^{\infty}P_n(x)Q_n(y),$$

where P_n and Q_n are Legendre Functions of degree n of the first and second kinds respectively.

10. *Applications to Elliptic and Hypergeometric Functions.*

Similarly taking the case of $\int_0^{\frac{\pi}{2}}(1-k^2\sin^2\phi)\,d\phi$

it is easily seen that

$$(1.10)\qquad \int_0^{\frac{\pi}{2}}(1-k^2\sin^2\phi)d\phi>\frac{\pi}{2}\,\frac{\int_0^{\frac{\pi}{2}}(1-k^2\sin^2\phi)^{\frac{1}{2}}d\phi}{\int_0^{\frac{\pi}{2}}\frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}}}$$

where

$$k^2=\vartheta_2^4(0|\tau)\Big/\vartheta_3^4(0|\tau)$$

i.e., k is the modulus of the elliptic function.

Thus we get

$$1-\frac{k^2}{2}>\frac{E}{K}\quad i.e.,\ \frac{F(-\frac{1}{2},\frac{1}{2};1;k^2)}{F(\frac{1}{2},\frac{1}{2};1;k^2)};$$

$$(1.11) \qquad \textit{i.e.,} \frac{\vartheta_4^4(0|\tau)+\vartheta_3^4(0|\tau)}{2.\,\vartheta_3^4(0|\tau)} > \frac{F(-\frac{1}{2}, \frac{1}{2}; 1; k^2)}{F(\frac{1}{2}, \frac{1}{2}; 1; k^2)}$$

also
$$1-\frac{k^2}{2} > \frac{\int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}(1-k^2u)^{\frac{1}{2}}\,du}{\int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}(1-k^2u)^{-\frac{1}{2}}\,du}$$

Again from the inequality

$$\int_0^{\frac{\pi}{2}} (1-k^2\sin^2\phi)^{\frac{1}{2}}d\phi \int_0^{\frac{\pi}{2}} \frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}} > \frac{\pi^2}{4}$$

follow the important properties

$$K\int_0^K dn^2u\,du > \frac{\pi^2}{4},$$

$$F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2).\,F(-\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) > 1,$$

and $\displaystyle\int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}(1-k^2u)^{-\frac{1}{2}}du \int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}(1-k^2u)^{\frac{1}{2}} > \pi^2.$

§ 2

Generalised Theorems.

11. The theorem of the previous section can easily be extended to the case of several functions.

The result corresponding to (1.1) can be put thus :—

THEOREM II. *If* $f_1(x), f_2(x), \ldots f_n(x), g_1(x), g_2(x) \ldots g_n(x)$, *be positive continuous functions of* x *in* $x_1 \leq x \leq x_2$, *then will*

$$(2.1) \qquad \int_{x_1}^{x_2}\frac{f_1(x)}{g_1(x)}\,dx \int_{x_1}^{x_2}\frac{f_2(x)}{g_2(x)}\,dx \ldots \int_{x_1}^{x_2}\frac{f_n(x)}{g_n(x)}\,dx$$

$$\leq (x_2-x_1)^n \frac{\int_{x_1}^{x_2} f_1(x)dx \ldots \int_{x_1}^{x_2} f_n(x)dx}{\int_{x_1}^{x_2} g_1(x)dx \ldots \int_{x_1}^{x_2} g_n(x)dx}$$

where some or all of the functions $\frac{f_1(x)}{g_1(x)}$, ... *and their corresponding ones among* $g_1(x),\ldots g_n(x)$ *increase or decrease simultaneously. But if some or all the members of the functions* $\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \ldots$ *increase or decrease, while the corresponding members of* $g_1(x)$, $g_2(x)\ldots$ *decrease or increase, we have the inequality in* (2.1) *reversed.*

In particular,

$$\int_{x_1}^{x_2}\frac{f_1(x)}{f_2(x)}\,dx\int_{x_1}^{x_2}\frac{f_2(x)}{f_3(x)}\,dx\ldots\int_{x_1}^{x_2}\frac{f_n(x)}{f_{n+1}(x)}\,dx \leq (x_2-x_1)^n\frac{\int_{x_1}^{x_2}f_1(x)dx}{\int_{x_1}^{x_2}f_{n+1}(x)dx} \tag{2.2}$$

where, either, $f_1(x), f_2(x)\ldots f_{n+1}(x)$ *are all positive increasing functions of* x *such that*

$$f_1(x) \succ f_2(x) \succ f_3(x) \succ \ldots \succ f_{n+1}(x),$$

or they are all positive decreasing functions such that

$$f_1(x) \prec f_2(x) \prec \ldots \prec f_{n+1}(x).$$

But the sign of inequality in (2.2) *will be reversed if the functions are all positive increasing functions of* x *such that*

$$f_1(x) \prec f_2(x) \prec \ldots \prec f_{n+1}(x)$$

or they are positive decreasing functions such that

$$f_1(x) \succ f_2(x) \succ \ldots \succ f_{n+1}(x),$$

These results follow easily from Theorem I. Since the integral of the quotient of any pair of functions varies as the quotient of the integral of the respective functions, the product of any number of such integrals must vary in the same manner, under the given conditions.

As in Theorem I, equality holds in (2.1) when $\frac{f_1}{g_1}, \frac{f_2}{g_2}, \ldots \frac{f_n}{g_n}$ are constants, or $g_1, g_2, \ldots g_n$ are constants, or some members of $\frac{f_1}{g_1}, \frac{f_2}{g_2}, \ldots$ and the remaining members of $g_1, g_2, \ldots$ are constants.

In the case of (2.2), equality holds when either $f_1, f_2, \ldots$ are all constants or their ratios $\frac{f_1}{f_2}, \frac{f_2}{f_3}$, are constants.

12. *Note 1.* When $\frac{f_1}{g_1} = \frac{f_2}{g_2} = \ldots = \frac{f_n}{g_n}$, we have (2.1) in the form

$$(2.3) \quad \left\{ \int_{x_1}^{x_2} \frac{f_1(x)}{g_1(x)}\, dx \right\}^n \leq \text{ or } \geq (x_2 - x_1)^n \left\{ \frac{\int_{x_1}^{x_2} f_1(x)\, dx}{\int_{x_1}^{x_2} g_1(x)\, dx} \right\}^n$$

which is the same thing as (1.1).

Note 2. If further $f_1(x), f_2(x) \ldots f_n(x)$ are all constants we have from (2.1)

$$(2.4.) \quad \left\{ \int_{x_1}^{x_2} \frac{dx}{g(x)} \right\}^n \geq (x_2 - x_1)^{2n} \frac{1}{\left\{ \int_{x_1}^{x_2} g(x)\, dx \right\}^n}$$

which evidently is the same as (1.6).

13. Next I give an important generalisation of Theorem I, from which will follow many interesting results and which includes two theorems of Dunkel * as particular cases. It is thus stated—

THEOREM III. *If $f(x)$ and $g(x)$ be two positive continuous functions of x defined in the interval $x_1 \leq x \leq x_2$, then will*

$$(2.5) \quad \left\{ \int_{x_1}^{x_2} \frac{f(x)}{g(x)}\, dx \right\}^n \leq \text{ or } \geq (x_2 - x_1)^n \frac{\int_{x_1}^{x_2} \{f(x)\}^n\, dx}{\int_{x_1}^{x_2} \{g(x)\}^n\, dx},$$

* O. Dunkel, *loc. cit.*

according as $\dfrac{f(x)}{g(x)}$ *and* $\{f(x)\}^{n-k}\ \{g(x)\}^{k}$ $(k=1, 2, 3, \ldots n)$ *both increase* *(or both decrease) or one increases and the other decreases.*

Proof :—

We have by Theorem I,

$$(2.6)\qquad \int_{x_1}^{x_2} \frac{f(x)}{g(x)}\, dx \leq \text{or} \geq (x_2-x_1)\frac{\displaystyle\int_{x_1}^{x_2} \{f(x)\}^n\, dx}{\displaystyle\int_{x_1}^{x_2} \{f(x)\}^{n-1} g(x)\, dx}$$

according as $\dfrac{f(x)}{g(x)}$ and $\{f(x)\}^{n-1} g(x)$ both increase (or both decrease), or one increases while the other decreases.

Similarly it can be easily seen that

$$\int_{x_1}^{x_2} \frac{f(x)}{g(x)} dx \leq \text{or} \geq (x_2-x_1)\frac{\displaystyle\int_{x_1}^{x_2} \{f(x)\}^{n-1} g(x)\, dx}{\displaystyle\int_{x_1}^{x_2} \{f(x)\}^{n-2}\{g(x)\}^2\, dx},$$

$$\int_{x_1}^{x_2} \frac{f(x)}{g(x)}\, dx \leq \text{or} \geq (x_2-x_1)\frac{\displaystyle\int_{x_1}^{x_2} \{f(x)\}^{n-2}\{g(x)\}^2\, dx}{\displaystyle\int_{x^1}^{x_2} \{f(x)\}^{n-3}\{g(x)\}^3 dx},$$

....

....

$$\int_{x_1}^{x_2} \frac{f(x)}{g(x)}\, dx \leq \text{or} \geq (x_2-x_1)\frac{\displaystyle\int_{x_1}^{x_2} f(x).\,\{g(x)\}^{n-1}\, dx}{\displaystyle\int_{x_1}^{x_2} \{g(x)\}^n\, dx}$$

holding under conditions similar to those of (2.6).

Hence combining these n inequalities we get

$$(2.5) \qquad \left\{ \int_{x_1}^{x_2} \frac{f(x)}{g(x)}\, dx \right\}^n \leq \text{ or } \geq (x_2 - x_1)^n \; \frac{\displaystyle\int_{x_1}^{x_2} \{f(x)\}^n\, dx}{\displaystyle\int_{x_1}^{x_2} \{g(x)\}^n\, dx},$$

according as $\dfrac{f(x)}{g(x)}$ and $\{f(x)\}^{n-k}\ \{g(x)\}^k \qquad (k=1,2,3,\ldots\ldots\ldots\ldots n)$ both increase (or both decrease), or one increases while the other decreases.

Equality holds when the ratio $\dfrac{f(x)}{g(x)}$ is constant.

14. *Note 1.* If we put $g(x)$ equal to some constant k in (2.5), we get

$$(2.7) \qquad \left\{ \int_{x_1}^{x_2} f(x)\, dx \right\}^n < (x_2 - x_1)^{n-1} \int_{x_1}^{x_2} \{f(x)\}^n\, dx.$$

This is a generalised from of Tchebycheff's inequality (1.5).

Note 2. Again taking $f(x)$ to be some constant k_1, we have

$$(2.8) \qquad \left\{ \int_{x_1}^{x^2} \frac{dx}{g(x)} \right\}^n > \frac{(x_2 - x_1)^{n+1}}{\displaystyle\int_{x_1}^{x_2} \{g(x)\}^n\, dx}$$

i.e.,

$$\left\{ \int_{x_1}^{x_2} \frac{dx}{g(x)} \right\}^n \int_{x_1}^{x_2} \{g(x)\}^n\, dx > (x_2 - x_1)^{n+1}.$$

This is the generalised form of (1.6).

In the inequalities (2.7) and (2.8), there is no restriction upon $f(x)$ and $g(x)$, so that they may be increasing or decreasing functions but necessarily positive.

These are the second and third theorems of Dunkel.

Note 3. The inequality (2.5) can be put as

$$(2.9) \qquad \left\{ \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \frac{f(x)}{g(x)}\, dx \right\}^n \leq \text{ or } \geq \frac{\displaystyle\frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \{f(x)\}^n\, dx}{\displaystyle\frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \{g(x)\}^n\, dx}.$$

This shows that *the nth power of the mean value of the quotient of two positive continuous functions is less than or greater than the quotient of the mean values of the nth power of the functions themselves, holding under the given conditions.*

In particular putting $n=2$, we have

$$(2.10)\qquad \left\{\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\frac{f(x)}{g(x)}dx\right\}^2 \leq \text{or} \geq \frac{\dfrac{1}{x_2-x_1}\displaystyle\int_{x_1}^{x_2}\{f(x)\}^2dx}{\dfrac{1}{x_2-x_1}\displaystyle\int_{x_1}^{x_2}\{g(x)\}^2dx}$$

which can be taken as an *analogue to Schwarz's inequality* which considers the product of two functions, whereas the inequality (2.10) considers the quotients.

§ 3.

Derivation of some well-known Inequalities.

15. In this section I propose to deduce certain well-known inequalities, namely those of Fujiwara, Hayashi, Schwarz and Cauchy with the help of the results obtained in the forogeing pages. All these inequalities are derived in the same manner.

16. *Deduction of Fujiwara's Inequality.**

We have from Theorem I

$$(3.1)\qquad \int_{x_1}^{x_2}\frac{\phi_1}{\phi_2}dx \leq (x_2-x_1)\frac{\displaystyle\int_{x_1}^{x_2}f_1\phi_1dx}{\displaystyle\int_{x_1}^{x_2}f_1\phi_2dx}$$

where f_1, f_2, ϕ_1 and ϕ_2 are any positive continuous functions of x such that $\dfrac{\phi_1}{\phi_2}$ and $f_1\phi_2$ both increase or both decrease in the given interval $x_1\leq x\leq x_2$.

* M. Fujiwara—"Ein von Brunn vermuteter Satz über konvexe Flächen und eine Verällgemeinerung der Schwarzschen und der Tchebycheffschen Ungleichungen für bestimmte Integrale," *Toho ku Mathematical Journal*, 13 (1918), pp. 228-235, and "Über eine Ungleichung für bestimmte Integrale." *Ibid*, 15 (1919), pp. 285-288,

Hence $$\frac{\int_{x_1}^{x_2} f_1\phi_1 dx}{\int_{x_1}^{x_2} f_1\phi_2 dx} \geq \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\frac{\phi_1}{\phi_2}\,dx.$$

Again $$\int_{x_1}^{x_2}\frac{\phi_1}{\phi_2} \geq (x_2-x_1)\frac{\int_{x_1}^{x_2} f_2\phi_1 dx}{\int_{x_1}^{x_2} f_2\phi_2 dx}$$

where one of the functions $\frac{\phi_1}{\phi_2}$ and $f_2\phi_2$ increases and the other decreases.

Hence combining these results we get

$$\text{(3.2)} \qquad \frac{\int_{x_1}^{x_2} f_1\phi_1 dx}{\int_{x_1}^{x_2} f_1\phi_2 dx} \geq \frac{\int_{x_1}^{x_2} f_2\phi_1 dx}{\int_{x_1}^{x_2} f_2\phi_2 dx}$$

and the combined condition shows that either $\frac{f_1}{f_2}$ and $\frac{\phi_1}{\phi_2}$ should both increase or both decrease to give the result (3.2), which is the condition formulated by Fujiwara.

The inequality (3.2) is reversed if one of the functions $\frac{f_1}{f_2}$ and $\frac{\phi_1}{\phi_2}$ be increasing while the other be decreasing.

Thus we have Fujiwara's inequality—

If f_1, f_2, ϕ_1 and ϕ_2 be any positive continuous functions of x defined in the interval $x_1 \leq x \leq x_2$, then will

$$\text{(3.3)} \qquad \int_{x_1}^{x_2} f_1\phi_1 dx \int_{x_1}^{x_2} f_2\phi_2 dx \geq \text{or} \leq \int_{x_1}^{x_2} f_1\phi_2 dx \int_{x_1}^{x_2} f_2\phi_1 dx$$

according as $\frac{f_1}{f_2}$ and $\frac{\phi_1}{\phi_2}$ both increase (or both decrease), or one increases while the other decreases.

17. *Derivation of Hayashi's inequality.*

The algebraical inequality corresponding to Fujiwara's was given by Hayashi.* It is very easily deduced from the algebraical one of my Theorem I given in para. 2. It is stated thus—

If a_ν, b_ν, c_ν and d_ν represent positive numbers for $\nu = 1, 2, 3 \ldots n$ then will,

$$(3.4) \qquad \sum_{\nu=1}^{n} a_\nu b_\nu \sum_{\nu=1}^{n} c_\nu d_\nu \geq \text{ or } \leq \sum_{\nu=1}^{n} a_\nu d_\nu \sum_{\nu=1}^{n} b_\nu c_\nu,$$

accoridng as $\dfrac{a_\nu}{c_\nu}$ and $\dfrac{b_\nu}{d_\nu}$ both increase (or both decrease), or one increases while the other decreases.

Proof :

We have

$$\sum_{\nu=1}^{n} \frac{b_\nu}{d_\nu} \leq n \frac{\sum_{\nu=1}^{n} a_\nu b_\nu}{\sum_{\nu=1}^{n} a_\nu d_\nu}, \text{ by (1.3)}$$

where $\dfrac{b_\nu}{d_\nu}$ and $a_\nu d_\nu$ both increase or both decrease.

Again $$\sum_{\nu=1}^{n} \frac{b_\nu}{d_\nu} \geq n \frac{\sum_{\nu=1}^{n} b_\nu c_\nu}{\sum_{\nu=1}^{n} c_\nu d_\nu}$$

where one of $\dfrac{b_\nu}{d_\nu}$ and $c_\nu d_\nu$ increases and the other decreases

Hence we get

$$\frac{\sum_{\nu=1}^{n} a_\nu b_\nu}{\sum_{\nu=1}^{n} a_\nu d_\nu} \geq \frac{\sum_{\nu=1}^{n} b_\nu c_\nu}{\sum_{\nu=1}^{n} c_\nu d_\nu}$$

i.e.,

$$(3.5) \qquad \sum_{\nu=1}^{n} a_\nu b_\nu \sum_{\nu=1}^{n} c_\nu d_\nu \geq \sum_{\nu=1}^{n} a_\nu d_\nu \sum_{\nu=1}^{n} b_\nu c_\nu$$

* T. Hayashi—"On Some Inequalities." *Rendiconti del Circolo Matematico di Palermo*, 44 (1920), pp. 336-340.

where $\frac{a_\nu}{c_\nu}$ and $\frac{b_\nu}{d_\nu}$ both increase or both decrease at the same time.

The inequality (3.5) is reversed if one of these increases while the other decreases. This proves the theorem.

From (3.4) also, by passing to the limit Fujiwara's inequality is obtained as in the case of Theorem I.

18. *Deduction of Schwarz's inequality.*

Take $f(x)$ and $g(x)$ to be positive continuous functions of x defined in $x_1 \leq x \leq x_2$.

Then $$\int_{x_1}^{x_2} \frac{g(x)}{f(x)}\, dx \geq (x_2 - x_1) \frac{\int_{x_1}^{x_2} f(x)\, g(x)\, dx}{\int_{x_1}^{x_2} \{f(x)\}^2\, dx}, \text{ by (1.1)}$$

where one of the functions $\frac{g(x)}{f(x)}$ and $\{f(x)\}^2$ increases and the other decreases.

Similarly $$\int_{x_1}^{x_2} \frac{g(x)}{f(x)}\, dx \leq (x_2 - x_1) \frac{\int_{x_1}^{x_2} \{g(x)\}^2\, dx}{\int_{x_1}^{x_2} f(x)\, g(x)\, dx},$$

where both the functions $\frac{g(x)}{f(x)}$ and $f(x)g(x)$ increase or decrease at the same time.

These twofold conditions, when combined, do away with any limitations on $\frac{f(x)}{g(x)}$ so that $f(x)$ and $g(x)$ are any two positive continuous functions of x, but opposite in sense. The same will also hold if they are of the same sense.

Hence we get Schwarz's inequality

$$\text{(3.6)} \quad \int_{x_1}^{x_2} f(x)\, g(x)\, dx \leq \left\{ \int_{x_1}^{x_2} \{f(x)\}^2\, dx \right\}^{\frac{1}{2}} \left\{ \int_{x_1}^{x_2} \{g(x)\}^2\, dx \right\}^{\frac{1}{2}}$$

Proceeding with this theorem to reverse the sign of inequality we land in an absurdity.

Similarly, from (1.3), we can easily deduce Cauchy's inequality. This is left to the reader.

Bull. Cal. Math. Soc., Vol. XXII, Nos. 2 and 3. (1930).

Review

An Introduction to the Geometry of n-dimensions

By

D. M. Y. Sommerville, Methuen & Co. Ltd., London, 1929
xviii + 196 pages.

The notion of higher spaces plays, at the present time, so useful and important a part in the natural extension of the bounds of knowledge in the various branches of higher mathematics that it is now regarded as an indispensable part thereof, intimately related to many other branches and with direct application to Mathematical Physics. Prior to 1914, with the exception of a few isolated papers, published at times in the different Journals, there were no regular treatises on the subject in the English language, and the absence was very keenly felt. In 1914, however, Profs. Manning* and Ganguli † published their books which dealt with only some particular aspects of the subject, and a great deal more remained, and still remains, to be done in this direction. The present *Introduction* is certainly an improvement upon the former publications, in so far as it approaches the subject of n-dimensional Geometry from various aspects—elementary, metrical, analytical and projective; but at the same time it must be said that a regular systematic treatment of the subject is still a desideratum.

In view of the complicated nature and lack of intuitive suggestiveness of the higher-space geometry, progress of investigation in this branch seems to have been greatly hampered, as compared with other branches of knowledge, and the subject may still be regarded in its infancy. Consequently, for obvious reasons an exhaustive treatment seems impracticable. Italian workers have no doubt done a great deal in this field, but the comparative apathy on the part of English authors is mainly, if not entirely, responsible for the absence of any such work in their own language. The present developed form of hypergeometry

* H. P. Manning—Geometry of Four dimensions, 1914, MacMillan Dompany New York.

† S. M. Ganguli—Analytical Geometry of Hyperspaces, Part I (1918) and Part II (1922), University of Calcutta, India.

is entirely the product of the giftedItalian school of geometers, headed by Veronese, Bertini, Segre and others, however much an English School may be desirable. Attempts seem to have been being made by a number of American workers to build up such a school and we are now glad to find Prof. Sommerville to join their rank. His publications on geometrical subjects have already acquired a name for him. His "Bibliography of non-Euclidean geometry including the theory of parallels, the foundations of Geometry and space of n-dimensions" (1911) clearly showed his keen interest in these subjects, and the appearance of the present work is highly welcome, while a more complete treatise would be extremely desirable.

The author opens by acquainting the reader with the nature of geometry and the fundamental ideas of higher dimensions based on a certain logical system of axioms. Certain principle are enunciated without attempt at their exposition. The first four chapters are devoted to the consideration of the fundamental ideas of incidence, parallelism, perpendicularity and angles between linear spaces. Orders of parallelism of linear spaces are explained with reference to spaces at infinity, while degrees of orthogonality are discussed with reference to a virtual Quadric in the $(n-1)$ flat at infinity. In Chapter IV, distances and angles between flat spaces are considered. In discussing angles between two planes in S_4 among other things it is stated that of the two angles ϕ_1, and ϕ_2, one is a maximum and the other a minimum and in § 10, they are again termed as minimum angles. The true nature of these angles should have been discussed and if no decision was possible, that should have been clearly stated. That both are minimum seems to be inconsistent with the fact that between two minima there is always a maximum, while the existence of a maximum requires elucidation with regard to its limit. The case of half orthogonal planes as stated on pages 34 and 45 should have been more clearly explained.

Further investigations in angle-concepts and isocline planes should have been incorporated.

In Chapter V projective properties of Quadrics and their linear spaces are studied analytically with the help of matrix notation. Chapter VI is devoted to the consideration of lines and planes in S_4 with the help of Cartesian Co-ordinates, and metrical geometry of S_4 is more fully discussed. The six co-ordinates of plane in S_4 have been conveniently used in calculating angles between two planes, and the process explained by an illustration. Lastly the condition for the intersection of a p-flat with a q-flat in a point is obtained. Considerable

space is devoted to polytopes with an account of regular polytopes in the last Chapter. Euler's Theorem and its relation to the angle-sums of a polytope are discussed in Chapter IX. In fact, these Chapters contain some of the elementary ideas of analysis situs. In Chapter VIII, expressions for the contents of a number of hyperspacial figures are given.

The author states in the preface that a complete systematic treatise is not attempted but certain representative topics are selected. The subject of hyperspace covers so extensive a field that the author is perfectly justified in selecting representative topics ; nevertheless the reviewer ventures to suggest that in view of the fact that the present work contains three chapters which discuss properties of linear spaces and as many as three chapters on polytopes, it might have been well to sacrifice a few of the duplications in order to include some of the topics that are omitted, namely, order, symmentry and rotation, specially in a space of four dimensions. Differential aspect of the subject has entirely been excluded.

At the end of each chapter we find a list of references both, as the author states, of original works in which the various theories were expounded, and of other recent work for a fuller account. As the subject is not yet in its fully developed form, it would perhaps be more convenient, if, following the continental authors, the references of important results were given in their appropriate places. In these lists a number of deplorable omissions is to be noticed, and this is certainly to be regarded as unfortunate specially on the part of the author of the *Bibliography*. Among others, one would sadly miss any mention of that great inspiring author Veronese and his classical work "Fondamenti di geometria, etc.", in which, it will be found, the essentials of almost every topic discussed in the work are given in some form or other.

Despite these minor blemishes, we prefer to end the review with deep appreciation to the author for his masterly production and for enriching the English language by bringing out this very useful treatise of its kind. The exposition on the whole is rigorous and painstaking and the matter presented is of considerable interest and unquestionable importance. A volume as attractive as this in form and in content is certainly destined to arouse interest in the subject and hence deserves an outstanding position in the current literature. We shall look forward with interest to the publication of a more complete work on the subject in the near future.

S. M. G.

Krishnakumari Ganesh Prasad Prize and Medal

FOR

RESEARCH IN THE HISTORY OF MATHEMATICS IN INDIA

BEFORE 1600 A.D.

Dr. Ganesh Prasad, Hardinge Professor of Higher Mathematics in the Calcutta University and President of the Calcutta Mathematical Society, has handed over to the Society $3\frac{1}{2}\%$ G. P. Notes of the face value of Rs. 1,400 for the creation of an endowment for the purpose of awarding a Prize and Medal in memory of his daughter. The Calcutta Mathematical Society has laid down the following rules for the award of the medal and prize :—

(1) A research prize and gold medal shall be instituted to be named Krishnakumari Ganesh Prasad Prize and Medal after the name of the donor's daughter.

(2) The prize and the medal shall be awarded every fifth year to the author of the best thesis embodying the result of original research or investigation in a topic connected with the history of Hindu Mathematics before 1600 A.D.

(3) The subject of the thesis shall be prescribed by the Council of the Calcutta Mathematical Society at least two years in advance.

(4) The last day of submitting the thesis for the award in a particular year shall be the 31st March of that year.

(5) The prize and the medal shall be open to competition to all nationals of the world without any distinction of race, caste or creed.

(6) A board of Honorary examiners consisting of (*i*) the President of the Society, (*ii*) an expert in the subject nominated by the donor, or after his death, such an expert nominated by the donor's heirs, and (*iii-v*) three experts in the subject elected by the Council of the Society, shall be appointed as soon as possible after the last day of receiving the theses.

(7) The recommendation of the Board of Examiners shall be placed before the next annual meeting of the Society and the decision of that meeting shall be final.

(8) Every candidate shall be required to submit three copies (type written) of his or her thesis.

(9) If in any year no theses is received or the theses submitted be pronounced by the Board of Examiners to be not of sufficient merit, a second prize or a prize in a second subject, or a prize of enhanced value, may be awarded in a subsequent year or years as the Council of the Calcutta Mathematical Society may determine.

(10) The thesis of the successful candidate shall be printed by the Society.

1

On the Summation of Infinite Series of Legendre's Functions

By

Ganesh Prasad.

The object of the present paper is to give a fairly long list of those infinite series of Legendre's functions which admit of being summed up into forms, compact and *free from the sign of integration.*

In addition to the known series, of which some are common to all well-known books on Spherical Harmonics and some are due to Bauer, Most, Heine, Routh, Darling or G. H. Stuart, the list contains a number of new results which were communicated by me to the Calcutta Mathematical Society nearly three years ago ; these new results appear in the list starred, and more or less full hints for obtaining them are given in the second part of the paper.

Unless the contrary is stated explicitly, the argument of every Legendre function is to be understood to be $x = \cos\theta$.

Part I.

(1) $$\frac{2}{\pi\sqrt{1-x^2}} = 1 + \sum_{n=1}^{\infty} (4n+1) . \left\{ \frac{1.3\ldots(2n-1)}{2.4\ldots 2n} \right\}^2 . P_{2n}$$

(*Bauer, Crelle's J., Bd. 56*)

(2) $$\frac{x}{\sqrt{1-x^2}} = \frac{\pi}{2} \left[\tfrac{3}{2} P_1 + \sum_{n=1}^{\infty} (4n+3) \left\{ \frac{1.3\ldots(2n-1)}{2.4\ldots 2n} \right\}^2 \times \frac{2n+1}{2n+2} . P_{2n+1} \right]$$

(*Ibid*)

(3) $$\frac{2}{\pi}\sqrt{1-x^2} = \tfrac{1}{2} P_0 - 5.\tfrac{1}{4}(\tfrac{1}{2})^2 P_2 - 9.\tfrac{3}{6}\left(\frac{1}{2.4}\right)^2 P_4 - 13.\tfrac{5}{8}\left(\frac{1.3}{2.4.6}\right)^2 P_6 - \cdots$$

(*Ibid*)

(4) $\frac{8}{\pi}\sin^{-1}x = 3P_1 + 7.(\frac{1}{4})^2 P_3 + 11.\left(\frac{1.3}{4.6}\right)^2 P_5 + \ldots\ldots$

(*Ibid*)

(5) $\log(1+x) = (\log 2 - 1)P_0 + \frac{3}{1.2}P_1$

$$-\frac{5}{2.3}P_2 + \frac{7}{3.4}P_3 - \ldots\ldots$$

(*Ibid*)

(6) $\left(\frac{1+x}{2}\right)^m = \frac{1}{1+m}P_0 + 3.\frac{m}{(m+1)(m+2)}P_1$

$$+5.\ \frac{m(m-1)}{(m+1)(m+2)(m+3)}P_2 + \ldots\ldots$$

(*Ibid*)

(7) $2e^{m\theta} = \frac{1+e^{m\pi}}{m^2+1^2}\left\{P_0 + \frac{m^2}{m^2+3^2}.\ 5\,P_2(x)\right.$

$$\left. + \frac{m^2(m^2+2^2)}{(m^2+3^2)(m^2+5^2)}.\ 9P_4(x) + \ldots\right\}$$

$$+\frac{1-e^{m\pi}}{m^2+2^2}\left\{3P_1(x) + \frac{m^2+1^2}{m^2+4^2}.\ 7\,P_3\,(x)\right.$$

$$\left. + \frac{(m^2+1^2)(m^2+3^2)}{(m^2+4^2)(m^2+6^2)}.\ 11P_5(x) + \ldots\right\}$$

(*Most, Crelle's J., Bd. 70*)

(8) $\frac{d^m P_n(x)}{dx^m} = \frac{2^m\Pi(n-\frac{1}{2})}{\Pi(n-m+\frac{1}{2})}\left\{(n-m+\tfrac{1}{2})P_{n-m} + \frac{m(n-m+\frac{1}{2})}{1.(n-\frac{1}{2})}\times\right.$

$$(n-m-\tfrac{3}{2})P_{n-m-2} + \frac{m(m+1).(n-m+\frac{1}{2})(n-m-\frac{1}{2})}{1.2.(n-\frac{1}{2})(n-\frac{3}{2})}\times$$

$$\left.(n-m-\tfrac{7}{2})P_{n-m-4} + \ldots\ldots\right\}$$

(*Ibid*)

(9) $$(1-x^2)^m = \frac{\Pi(-\frac{1}{2})\Pi(m)}{\Pi(m+\frac{1}{2})}\left\{\tfrac{1}{2} - \tfrac{5}{2}.\frac{1.m}{2.(m+\frac{3}{2})}\,P_2 + \tfrac{9}{2}.\frac{1.3\ldots m(m-1)}{2.4.(m+\frac{3}{2})(m+\frac{5}{2})}\,P_4 + \ldots\right\}$$

(*Ibid*)

(10) $$E(k) = \frac{\pi^2}{8} + \frac{\pi^2}{4}\sum_1^\infty \frac{(-1)^{n+1}(4n+1)}{(2n-1)(2n+2)}\left\{\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\right\}^3 . P_{2n}(k')$$

(*Hargreaves, Mess. Math.*, 1897).†

(11) $$(1+kx^2)^{-\gamma} = \sum_0^\infty b_{2n}\, P_{2n}(x),$$

where

(*Heine, Handbuch, Bd. 1*)

$$b_{2n} = \frac{(2n)!\,(-k)^n}{1.3.5\ldots(4n-1)}\;\frac{\gamma(\gamma+1)\ldots(\gamma+n-1)}{n\,!}\times F(n+\tfrac{1}{2},\, n+\gamma,\, 2n+\tfrac{3}{2}, -k)$$

(12) $$\cos m\left(\frac{\pi}{2}-\theta\right) = \frac{\cos m\frac{\pi}{2}}{1-m^2}\left\{1.P_0 + 5.\frac{m^2}{m^2-3^2}\,P_2 + 9.\frac{m^2(m^2-2^2)}{(m^2-3^2)(m^2-5^2)}\,P_4 + \ldots\ldots\right\}$$

(13) $$\sin m\left(\frac{\pi}{2}-\theta\right) = \frac{\sin m\frac{\pi}{2}}{2^2-m^2}\left\{3.P_1 + 7.\frac{m^2-1^2}{m^2-4^2}\,P_3 + 11.\frac{(m^2-1^2)(m^2-3^2)}{(m^2-4^2)(m^2-6^2)}\,P_5 + \ldots\ldots\right\}$$

(14) $$2.\frac{3.5\ldots(2m+1)}{2.4\ldots 2m}\cos m\theta = (2m+1)P_m + (2m-3)\times \frac{m^2-(m+1)^2}{m^2-(m-2)^2}P_{m-2} + (2m-7).\frac{\{m^2-(m+1)^2\}\{m^2-(m-1)^2\}}{\{m^2-(m-2)^2\}\{m^2-(m-4)^2\}}\,P_{m-4} + \ldots$$

† Prof. Hardy gives this result in *Proc. L. M. S.*, 1924, p. XII, without mentioning Hargreaves.

(15) $$\frac{4}{\pi}\cdot\frac{2.4\ldots(2m-2)}{1.3\ldots(2m-3)}\sin m\,\theta=(2m-1)P_{m-1}$$

$$+(2m+3).\frac{(m-1)^2-m^2}{(m+2)^2-m^2}P_{m+1}$$

$$+(2m+7).\frac{\{(m-1)^2-m^2\}\{(m+1)^2-m^2\}}{\{(m+2)^2-m^2\}\{(m+4)^2-m^2\}}P_{m+3}$$

$$+\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

(16) $$\frac{1-a^2}{(1-2ax+a^2)^{\frac{3}{2}}}=\sum_{n=0}^{\infty}(2n+1)a^n.\,P_n(x).$$

(17) $$\frac{3(1-a^2)^3}{(1-2ax+a^2)^{\frac{5}{2}}}=\sum_{0}^{\infty}\{(4n^2+8n+3)-(4n^2-1)a^2\}a^nP_n(x)$$

(18) $$\frac{1.3.5.(1-a^2)^5}{(1-2ax+a^2)^{\frac{7}{2}}}=\sum_{0}^{\infty}(2n+1)\{(2n+5)(2n+3)-2a^2(2n-3)\times$$
$$(2n+5)+a^4(2n-3)(2n-1)\}a^nP_n$$

(19) $$\frac{1.3.5\ldots(2m-1).(1-a^2)^{2m-1}}{(1-2ax+a^2)^{\frac{2m+1}{2}}}\sum_{0}^{\infty}(2n+1)A_na^nP_n(x),\text{ where}$$

$$A_n=2^{m-1}\left(a^2\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^{m-1}.\,x^{-\frac{1}{2}(2n-2m+3)}y^{\frac{1}{2}(2n+2m-1)},$$

and where x, y are to be replaced by 1 after the differentiation has been performed.

(*Routh, Proc. L. M. S., Ser. I, Vol. 26*)

(20) $$\log\left(1+\operatorname{cosec}\frac{\theta}{2}\right)=\sum_{0}^{\infty}\frac{P_n}{n+1}.$$

(21) $$-\log\left\{\sin\frac{\theta}{2}\cdot\left(1+\sin\frac{\theta}{2}\right)\right\}=\sum_{1}^{\infty}\frac{P_n(\cos\theta)}{n}$$

(22) $K=\sum_{n=0}^{\infty}\frac{2P_n}{2n+1}$, K standing for the complete elliptic integral of the first kind with modulus $\cos\frac{\theta}{2}$ (*Darling and G. H Stuart*).†

† See Darling's paper in *Quarterly J. of Math.*, Vol. 49.

(23) $$1-2\log\left(1+\sin\frac{\theta}{2}\right)=\sum_{1}^{\infty}\frac{P_n(\cos\theta)}{n(n+1)}.$$

(24)* $$-\log\frac{1-ax+\sqrt{1-2ax+a^2}}{2}=\sum_{1}^{\infty}\frac{a^n P_n(x)}{n}$$

(25)* $$\log 2-\tfrac{1}{2}\log(1+x)-\log(1+\sqrt{x})=\sum_{1}^{\infty}\frac{(-1)^m P_{2m}(x)}{2m}.$$

(26)* $$\tan^{-1}\sqrt{x}=\sum_{0}^{\infty}\frac{(-1)^m P_{2m+1}(x)}{2m+1}.$$

(27)* $$-\log\frac{1-ax+\sqrt{1-2ax+a^2}}{2}+1$$

$$-\frac{1}{a}\log\left\{\frac{a-x+\sqrt{1-2ax+a^2}}{1-x}\right\}=\sum_{1}^{\infty}\frac{a^n P_n(x)}{n(n+1)}.$$

(28)* $$1+\log 2-\tfrac{1}{2}\log(1+x)-\log(1+\sqrt{x})-\tan^{-1}\frac{1}{\sqrt{x}}$$

$$=\sum_{1}^{\infty}\frac{(-1)^m P_{2m}(x)}{2m(2m+1)}.$$

(29)* $$\tan^{-1}\sqrt{x}+\tfrac{1}{2}\log(1+x)+\log\frac{1-\sqrt{x}}{1-x}$$

$$=\sum_{0}^{\infty}\frac{(-1)^m P_{2m+1}(x)}{(2m+1)(2m+2)}.$$

(30)* $$\sqrt{1-2ax+a^2}+x\log\frac{a-x+\sqrt{1-2ax+a^2}}{1-x}-1$$

$$=\sum_{0}^{\infty}\frac{a^{n+2}P_n(x)}{n+2}.$$

(31)* $$\frac{a}{2}\sqrt{1-2ax+a^2}+\tfrac{3}{2}x(\sqrt{1-2ax+a^2}-1)$$

$$+(\tfrac{3}{2}x^2-\tfrac{1}{2})\log\frac{a-x+\sqrt{1-2ax+a^2}}{1-x}$$

$$=\sum_{0}^{\infty}\frac{a^{n+3}P_n(x)}{n+3}.$$

$$(32)^* \quad \frac{a^2}{3}\sqrt{1-2ax+a^2}+\frac{5}{3.2}\,xa\sqrt{1-2ax+a^2}$$

$$+\left(\tfrac{5}{3}.\tfrac{3}{2}x^2-\tfrac{2}{3}\right)\left(\sqrt{1-2ax+a^2}+x\log\frac{a-x+\sqrt{1-2ax+a^2}}{1-x}-1\right)$$

$$-\tfrac{5}{3}.\tfrac{1}{2}\log\frac{a-x+\sqrt{1-2ax+a^2}}{1-x}$$

$$=\sum_0^\infty \frac{a^{n+4}P_n(x)}{n+4}.$$

$$(33)^* \quad \sqrt{2}\,\sqrt{1-x}+x\log\frac{1-x+\sqrt{2}\,\sqrt{1-x}}{1-x}-1=\sum_0^\infty\frac{P_n(x)}{n+2}$$

$$(34)^* \quad \frac{1}{\sqrt{2}}\sqrt{1-x}+\tfrac{3}{2}x(\sqrt{2}\,\sqrt{1-x}-1)+\frac{3x^2-1}{2}$$

$$\times\log\frac{1-x+\sqrt{2}\,\sqrt{1-x}}{1-x}=\sum_0^\infty\frac{P_n(x)}{n+3}.$$

$$(35)^* \quad \tfrac{1}{2}-\cos\frac{\theta}{2}\sin\theta=\sum_0^\infty\frac{P_2P_n}{n+2}-\sum_0^\infty\frac{P_1P_n}{n+3}$$

$$(36)^* \quad \tfrac{1}{3}\sqrt{2}\,\sqrt{1-x}+\frac{5}{3.2}\,x\sqrt{2}\,\sqrt{1-x}+\left(\tfrac{5}{2}x^2-\tfrac{2}{3}\right)$$

$$\times\left(\sqrt{2}\,\sqrt{1-x}+x\log\frac{1-x+\sqrt{2}\,\sqrt{1-x}}{1-x}-1\right)$$

$$-\frac{5}{3.2}\log\frac{1-x+\sqrt{2}\,\sqrt{1-x}}{1-x}=\sum_0^\infty\frac{P_n(x)}{n+4}.$$

$$(37)^* \quad 2\sum_0^\infty P_n\,\frac{2.4\ldots2n}{3\cdot5\ldots(2n+1)}=\sqrt{\frac{1}{2\sin\frac{\theta}{2}}}\,cn^{-1}\left\{\frac{2\sin\frac{\theta}{2}-1}{2\sin\frac{\theta}{2}+1},\right.$$

$$\left.\sqrt{\left(\frac{1+\sin\frac{\theta}{2}}{2}\right)}\right\}$$

$$(38)^* \quad \sum_0^\infty P_n(x).\,\frac{r^{2n+1}}{2n+1}=K-\tfrac{1}{2}cn^{-1}\left(\frac{r^2-1}{r^2+1},\cos\frac{\theta}{2}\right)$$

(39)* $\sum_{0}^{\infty} P_n(x) . \frac{r^{4n+1}}{4n+1} = \frac{1}{2}(U+V),$

where

$$U = -\frac{1}{\alpha} \operatorname{sn}^{-1}\left(\frac{\alpha}{r+\frac{1}{r}}, \frac{\beta}{\alpha} \right),$$

$$V = \frac{1}{\alpha_1} \operatorname{tn}^{-1}(\infty) - \frac{1}{\alpha_1} \left\{ \operatorname{tn}^{-1} \left(\frac{\frac{1}{r} - r}{\beta_1}, \sqrt{1 - \frac{\beta_1^2}{\alpha_1^2}} \right) \right\}$$

$$\alpha = \sqrt{2+2\cos\frac{\theta}{2}}, \quad \beta = \sqrt{2-2\cos\frac{\theta}{2}},$$

$\alpha_1 = \alpha, \ \beta_1 = \beta.$

(40)* $\sum_{0}^{\infty} P_n(x) . \frac{r^{4n+3}}{4n+3} = \frac{1}{2}(U-V)$

(41)* $$\sum_{0}^{\infty} \frac{P_n}{4n+1} = -\frac{1}{2\sqrt{2+2\cos\frac{\theta}{2}}} \times$$

$$\operatorname{sn}^{-1}\left(\frac{\sqrt{2+2\cos\frac{\theta}{2}}}{2}, \frac{\sqrt{1-\cos\frac{\theta}{2}}}{\sqrt{1+\cos\frac{\theta}{2}}} \right)$$

$$+\frac{K}{2} . \frac{1}{\sqrt{2+2\cos\frac{\theta}{2}}}$$

(42)* $$\sum_{0}^{\infty} \frac{P_n}{4n+3} = -\frac{1}{2\sqrt{2+2\cos\frac{\theta}{2}}} \times$$

$$\left\{ \operatorname{sn}^{-1}\left(\frac{\sqrt{2+2\cos\frac{\theta}{2}}}{2}, \frac{\sqrt{1-\cos\frac{\theta}{2}}}{\sqrt{1-\cos\frac{\theta}{2}}} \right) + K \right\}.$$

Part II.

Result (24).

$$\sum_{0}^{\infty} a^n \mathrm{P}_n(x) = (1-2ax+a^2)^{-\frac{1}{2}},$$

$$i.e., \sum_{1}^{\infty} \frac{a^n \mathrm{P}_n}{a} = \frac{1}{a}\{(1-2ax+a^2)^{-\frac{1}{2}}-1\}$$

Integrate both the sides of the above with respect to a, then

$$\sum_{1}^{\infty} \frac{a^n \mathrm{P}_n}{n} = \int_0^a \frac{1}{a}\{(1-2ax+a^2)^{-\frac{1}{2}}-1\}da$$

$$= -\log \frac{1-ax+\sqrt{1-2ax+a^2}}{2}$$

Results (25) *and* (26).

Put $a=i$ in (24) and equate the real part on the left to the real part on the right, then we have (25) ; similarly, equating the imaginary parts, we have (26).

Result (27).

From (24), by integrating both the sides with respect to a, we have

$$\sum_{1}^{\infty} \frac{a^n \mathrm{P}_n}{n(n+1)} = -\int_0^a \log \frac{1-ax+\sqrt{1-2ax+a^2}}{2} da$$

$$=1-\log \frac{1-ax+\sqrt{1-2ax+a^2}}{2} - \frac{1}{a}\log\left\{\frac{a-x+\sqrt{1-2ax+a^2}}{1-x}\right\}.$$

Results (28) *and* (29).

Equating the real part on the left to the real part on the right after putting $a=i$, we get (28) ; similarly, by equating the imaginary parts, we get (29).

Result (30).

Multiply both the sides of

$$\sum_0^\infty a^n P_n = (1-2ax+a^2)^{-\frac{1}{2}}$$

by a, and integrate with respect to a.

Then

$$\sum_0^\infty \frac{a^{n+2}P_n}{n+2} = \int_0^a \frac{ada}{\sqrt{1-2ax+a^2}} = \sqrt{1-2ax+a^2} - 1$$

$$+x \int_0^a \frac{da}{\sqrt{1-2ax+a^2}}$$

$$= \sqrt{1-2ax+a^2} + x \log \frac{a-x+\sqrt{1-2ax+a^2}}{1-x} - 1.$$

Results (31) and (32).

These follow from the following general procedure :

$$\sum_0^\infty a^n P_n(x) = (1-2ax+a^2)^{-\frac{1}{2}}$$

Therefore, multiplying both the sides by a^k and integrating term by term, we have

$$\sum_0^\infty \frac{a^{n+k+1}}{n+k+1} P_n(x) = \int_0^a a^k(1-2ax+a^2)^{-\frac{1}{2}} da$$

$$= \frac{a^{k-1}}{k}\sqrt{1-2ax+a^2} + \frac{2k-1}{k(k-1)} x\, a^{k-2}\sqrt{1-2ax+a^2}$$

$$+\left(\frac{2k-1}{k}\cdot\frac{2k-3}{k-1}x^2-\frac{k-1}{k}\right)\int_0^a\frac{a^{k-2}da}{\sqrt{1-2ax+a^2}}$$

$$-\frac{2k-1}{k}\cdot\frac{k-2}{k-1}\int_0^a\frac{a^{k-3}da}{\sqrt{1-2ax+a^2}}$$

(by the ordinary processes of Integral Calculus)

Thus, for $k=2$, we have (31) ; similarly, for $k=3$, we have (32).

Results (33) to (36).

These follow from the preceding three results by putting $a=1$.

Result (37).

Multiply both the sides of

$$\sum_0^\infty a^n P_n=(1-2ax+a^2)^{-\frac{1}{2}}$$

by $\dfrac{1}{\sqrt{1-a}}$ and integrate with respect to a, then we have

$$\sum_0^\infty P_n(x)\int_0^1\frac{a^n}{\sqrt{1-a}}da,\ i.e.,\ \sum_0^\infty P_n.2.\frac{2.4\ldots2n}{3.5\ldots(2n+1)}$$

$$=\int_0^1\frac{da}{\sqrt{1-a}\sqrt{1-2ax+a^2}}=\frac{1}{\sqrt{2}\sin\frac{\theta}{2}}\operatorname{cn}^{-1}\left\{\frac{2\sin\frac{\theta}{2}-1}{2\sin\frac{\theta}{2}+1},\ \sqrt{\left(\frac{1+\sin\frac{\theta}{2}}{2}\right)}\right\}$$

(by a suitable transformation as e.g., in Greenhill's *Elliptic Functions*, p. 40.)

Result (38).

In

$$\sum_{0}^{\infty} a^n P_n(x) = (1-2ax+a^2)^{-\frac{1}{2}},$$

put $a=r^k$. Then

$$\sum_{0}^{\infty} r^{kn} P_n = (1-2r^k x+r^{2k})^{-\frac{1}{2}}.$$

Multiply both the sides of the above by r^a and integrate with respect to r, then

$$\sum_{0}^{\infty} \frac{r^{kn+a+1}}{kn+a+1} P_n = \int_0^r (1-2r^k x+r^{2k})^{-\frac{1}{2}} r^a \, dr \tag{A}$$

As a special case, take $k=2$, $a=0$, then the above becomes

$$\sum_{0}^{\infty} \frac{r^{2n+1}}{2n+1} P_n = \int_0^r (1-2r^2 x+r^4)^{-\frac{1}{2}} dr$$

$$= K - \tfrac{1}{2} \operatorname{cn}^{-1}\left(\frac{r^2-1}{r^2+1}, \cos \frac{\theta}{2} \right),$$

[by a formula in Greenhill's book, p, 60.]

Result (39).

Take $k=4$, $a=0$, in (A) then

$$\sum_{0}^{\infty} \frac{r^{4n+1}}{4n+1} P_n = \int_0^r (1-2r^4 x+r^8)^{-\frac{1}{2}} dr = \tfrac{1}{2}(U+V)$$

[See Greenhill's book, p. 162]

Result (40).

Take $k=4$, $a=2$ in (A), then the result follows as above.

Results (41) *and* (42).

These follow from (39) and (40) respectively, on putting $r=1$.

2

A Hydrodynamical Problem : Motion due to a System of Bodies

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(Communicated by Prof B. M. Sen)

Introduction.

The motion of translation of a single circular disc * in an infinite liquid is well-known; the translation of two circular discs has been considered. The method followed is that of Dr. J. W. Nicholson † in his Memoir on the electrification of two circular discs. An Integral equation has been formed of Dr. Nicholson's type, which has been solved approximately.

The steady rotation of a single circular disc in a viscous liquid being considered by Dr. G. B. Jeffery, ‡ we have contemplated the rotation of two circular discs, equal and un-equal. In the case of un-equal discs, a method of approximation has been followed, while in the case of equal circles, the method of approximation as well as that of Integral Equation has been introduced.

The Fourier-Bessel Integral of spheroidal harmonics has been necessary; but Dr. Nicholson's method of deriving the integral, as applied to spheroidal harmonics of zonal types, seems to be unsuitable when spheriodal harmonics of associated types are concerned. By using the some of his formulas, as also otherwise, we have succeeded in having corresponding Fourier-Bessel Integral. A general formula which includes Bauer's § formula as a special case, has been obtained.

I take this opportunity of expressing my sense of gratefulness to Prof. B. M. Sen, M.A. (Cantab.), Presidency College, Calcutta, for the interest he has taken.

* Lamb, Hydrodynamics.

† *Phil. Trans. Royal Soc.,* Vol. 224 (1924).

‡ *Proc. Lond. Math. Soc.,* 1915, p. 327.

§ *Münchener Sitzungsber.,* v, 1875, p. 263.

Section I.

Translation of two circular discs.

Preliminary Statements.

Two oblate spheroids of different radii of confocality, a and b are contemplated, having the same axis of revolution which is taken as the z-axis; (ζ, μ, ω) and (ζ_1, μ_1, ω) denote the oblate spheroidal co-ordinates of a point in space with different origins O (the centre of the oblate spheroid, b) and O_1 (the centre of the oblate spheroid, a) respectively; (z,ρ,ω) and (z_1,ρ_1,ω) denote the corresponding cylindrical co-ordinates, the z-axes being measured in opposite senses OO_1 and O_1O respectively. We have with the usual idea, the geometrical relations

$$\rho=a\sqrt{1-\mu_1^2}\ \sqrt{\zeta_1^2+1} = b\sqrt{1-\mu^2}\ \sqrt{\zeta^2+1} \text{ and } b\mu\zeta=c-a\mu_1\zeta_1 \ldots(1)$$

the spheroids are supposed to move with velocities v_1 and v_2 along the common axis of revolution.

Fourier-Bessel Integral of a Spheroidal Harmonic of Associated Type.

Deriving the analogue of the well-known Formula *

$$P_n^m(\mu)Q_n^m(\zeta)=\frac{1}{\pi}\cdot\frac{(n+m)!}{(n-m)!}\int_0^\pi Q_n\left[\mu\zeta+\sqrt{1-\mu^2}\ \sqrt{1-\zeta^2}\cos u\right]\cos mu du$$

in q-Functions, we have,

$$P_n^m(\mu)q_n^m(\zeta)=\frac{i^m}{\pi}\cdot\frac{(n+m)!}{(n-m)!}\int_0^x q_n\left[\mu\zeta-i\sqrt{1-\mu^2}\ \sqrt{\zeta^2+1}\cos u\right]\cos mu\, du$$

$$=\frac{i^m}{\pi}\cdot\frac{(n+m)!}{(n-m)!}\int_0^\pi q_n\left(\frac{z-i\rho\cos u}{a}\right)\cos mu\, du$$

$$=\frac{i^m}{\pi}\cdot\frac{(n+m)!}{(n-m)!}\sqrt{\frac{\pi a}{2}}\int_0^\pi\int_0^\infty e^{-\lambda(z-i\rho\cos u)}J_{n+\frac{1}{2}}(\lambda a)\frac{d\lambda}{\sqrt{\lambda}}\cos mu\, du \dagger$$

* Jeffery, *Proc. Edin. Math. Soc.*, p. 119, Vol. 33.

† Nicholson, *Phil. Trans. Royal. Soc.*, Vol. 224, 1924, p. 56.

$$= \frac{i^m}{\pi} \cdot \frac{(n+m)!}{(n-m)!} \sqrt{\frac{\pi a}{2}} \int_0^\infty e^{-\lambda z} J_{n+\frac{1}{2}}(\lambda a) \frac{d\lambda}{\sqrt{\lambda}}.$$

$$\times \int_0^\pi \cos mu \; du \left[J_0(\lambda\rho) + 2 \sum_{s=1}^{\infty} i^s J_s(\lambda\rho) \cos su \right]^*$$

$$= (-)^m \frac{(n+m)!}{(n-m)!} \cdot \sqrt{\frac{\pi a}{2}} \cdot \int_0^\infty e^{-\lambda z} J_m(\lambda\rho) \; J_{n+\frac{1}{2}}(\lambda a) \frac{d\lambda}{\sqrt{\lambda}}. \quad \ldots(2)$$

Transformation of a Spheroidal Harmonic of Associated Type corresponding to a change of origin.

$P_n^m(\mu_1) q_n^m(\zeta_1)$

$$= \frac{i^m}{\pi} \cdot \frac{(n+m)!}{(n-m)!} \int_0^\pi q_n \left[\mu\zeta_1 - i\sqrt{1-\mu_1^2} \; \sqrt{\zeta_1^2+1} \cos u \right] \cos mu \; du.$$

$$= \frac{i^m}{\pi} \cdot \frac{(n+m)!}{(n-m)!} \int_0^\pi q_n \left[\frac{c}{a} - \frac{b}{a} \mu\zeta - \frac{ib}{a} \sqrt{1-\mu^2} \; \sqrt{\zeta^2+1} \cos u \right]$$

$$\times \cos mu \; du$$

$$= \frac{i^m}{\pi} \cdot \frac{(n+m)!}{(n-m)!} \sum_{r=0}^{\infty} 2^n \frac{(n+r)!\,(n+2r)!\,(-)^r}{r!\,(2n+2r+1)!}$$

$$\int_0^\pi \frac{\cos mu \; du}{\left[\frac{c}{a} + \frac{b}{a} \mu\zeta - \frac{bi}{a} \sqrt{1-\mu^2} \; \sqrt{\zeta^2+1} \cos u \right]^{n+2r+1}}$$

$$= \frac{i^m}{\pi} \cdot \frac{(n+m)!}{(n-m)!} \sum_{r=0}^{\infty} 2^n \frac{(-)^{n+r} (n+r)!\, D^{n+2r}}{r!\,(2n+2r+1)!}$$

$$\times \frac{a}{b} \int_0^\pi \frac{\cos mu \; du}{c/b - \zeta\mu - i\sqrt{1-\mu^2}\sqrt{\zeta^2+1} \cos u},$$

$$\left[D = a \frac{\partial}{\partial c} \right.$$

* Watson, The Theory of Bessel Function, p. 22.

$$= \frac{i^m}{\pi} \cdot \frac{(n+m)\,!}{(n-u)\,!} (-)^n \sqrt{\frac{\pi}{2\mathrm{D}}}\ \mathrm{J}_{n+\frac{1}{2}}(\mathrm{D}) \frac{a}{b} \int_0^\pi i \sum_{r=0}^{\infty}$$

$$(2r+1)\mathrm{Q}_r(ic/b)\mathrm{P}_r\{i\mu\zeta - \sqrt{1-\mu^2}\,\sqrt{\zeta^2+1}\cos u\} \cos mu\, du \ *$$

$$= \frac{(n+m)\,!}{(n-m)} (-)^n \sqrt{\frac{\pi}{2\mathrm{D}}}\, \mathrm{J}_{n+\frac{1}{2}}(\mathrm{D}) \frac{a}{b} \sum_{r=m}^{\infty} (-)^m (2r+1) q_r(c/b)$$

$$\frac{(r-m)\,!}{(r+m)\,!} \mathrm{P}_r^m(\mu) p_r^m(\zeta) \ \dagger$$

$$= \sum_{r=m}^{\infty} {}_1\omega_n^m(r, c, a, b)(2r+1)\mathrm{P}_r^m(\mu)p_r^m(\zeta), \quad \ldots \quad (3)$$

where ${}_1\omega_n^m(r, c, a, b)$

$$= (-)^{n+m} \frac{a}{b} \frac{(r-m)\,!}{(r+m)\,!} \frac{(n+m)\,!}{(n-m)\,!} \sqrt{\frac{\pi}{2\mathrm{D}}}\ \mathrm{J}_{n+\frac{1}{2}}(\mathrm{D})\ q_r(c/b) \ldots \quad (4)$$

When $m=0$, we have,

$$\mathrm{P}_n(\mu_1)q_n(\zeta_1)$$

$$= \sum_{r=0}^{\infty} (2r+1)\ {}_1w_n^0(r, c, a, b)\ \mathrm{P}_r(\mu)p_r(\zeta), \quad \ldots \quad (4a)$$

where ${}_1\omega_n^0(r, c, a, b)$

$$= (-)^n \frac{a}{b} \sqrt{\frac{\pi}{2\mathrm{D}}}\ \mathrm{J}_{n+\frac{1}{2}}(\mathrm{D})\ q_r\left(\frac{c}{b}\right)$$

$$= (-)^n \sqrt{\frac{a}{b}}\ \frac{\pi}{2} \cdot \frac{\mathrm{J}_{n+\frac{1}{2}}\left(a\frac{\partial}{\partial c}\right)}{\sqrt{\frac{\partial}{\partial c}}} \int_0^\infty e^{-\lambda c}\ \mathrm{J}_{r+\frac{1}{2}}(\lambda b) \frac{d\lambda}{\sqrt{\lambda}}$$

$$= \frac{\pi}{2} \sqrt{\frac{a}{b}} \int_0^\infty e^{-\lambda c}\ \mathrm{J}_{n+\frac{1}{2}}(\lambda a)\ \mathrm{J}_{r+\frac{1}{2}}(\lambda b) \frac{d\lambda}{\lambda} \quad \ldots \quad (4b)$$

* Whittaker & Watson, Mod. Analysis, 3rd. ed., p. 322, Art. 15, 4.

† Blades, *Proc. Edin. Math. Soc.*, Vol. 33, p. 68.

Similarly, $P_n^m(\mu)q_n^m(\zeta)$

$$= \sum_{r=0}^{\infty} (2r+1)\,{}_2\omega_n^m(r, c, a, b,)\, P_r^m(\mu_1)p_r^m(\zeta_1),$$

where ${}_2\omega_n^m(r, c, a, b)$

$$= (-)^{n+m}\frac{b}{a}\cdot\frac{(r-m)\,!}{(r+m)\,!}\;\frac{(n+m)\,!}{(n-m)\,!}\;\sqrt{\frac{\pi}{2D'}}\;J_{n+\frac{1}{2}}(D')\;q_r\left(\frac{c}{a}\right),$$

$$D'=b\frac{\partial}{\partial c} \qquad \ldots \quad (5)$$

When $m=0$, we have,

$$P_n(\mu)q_n(\zeta) = \sum_{r=0}^{\infty} (2r+1)\;{}_2\omega_n^0(r, c, a, b)P_r(\mu_1)p_r(\zeta_1), \qquad \ldots \quad (5a)$$

where ${}_2\omega_n^0(r, c, a, b,)$

$$= (-)^n\frac{b}{a}\cdot\sqrt{\frac{\pi}{2D'}}\;J_{n+\frac{1}{2}}(D')\;q_r\left(\frac{c}{a}\right)$$

$$= \frac{\pi}{2}\sqrt{\frac{b}{a}}\int_0^{\infty} J_{n+\frac{1}{2}}(\lambda a)\;J_{r+\frac{1}{2}}(\lambda b)\;e^{-\lambda c}\frac{d\lambda}{\lambda} \qquad \ldots \quad (5b)$$

A General Formula (Application of Transformation).

Obtaining the transformation of a spheroidal Harmonic in two different ways, a general formula can be deduced, which includes Bauer's * formula as a special case. But it is derived from the equality of two integrals. Assuming, for simplicity, the radii of confocality to be equal to unity and the geometrical relations such as those previously stated, we have,

$P_n^m(\mu)Q_n^m(\zeta)$

$$= \frac{1}{\pi}\frac{(n+m)\,!}{(n-m)\,!}\int_0^{\pi} Q_n[\mu\zeta+\sqrt{1-\mu^2}\;\sqrt{1-\zeta^2}\cos u]\cos mu\,du$$

$$= \frac{1}{\pi}\frac{(n+m)\,!}{(n-m)\,!}\sum_0^{\infty} 2^n\frac{(n+r)\,!\,(n+2r)\,!}{r\,!\,(2n+2r+1)\,!}$$

$$\times\int_0^{\pi}\frac{\cos mu\,du}{[c-\mu_1\zeta_1+\sqrt{1-\mu_1^2}\;\sqrt{1-\zeta_1^2}\cos u]^{n+2r+1}}$$

* *Münchener Sitzungsber.*, v (1875), p. 263.

$$=\frac{1}{\pi}\frac{(n+m)!}{(n-m)!}\sum_0^\infty(-2)^n\frac{(n+r)!}{r!\,(2n+2r+1)!}\,\mathrm{D}^{n+2r}\,(-i)$$

$$\times\int_0^\pi\frac{\cos mu\,du}{-ic+i\mu_1\zeta_1-i\sqrt{1-\mu_1^2}\,\sqrt{1-\zeta_1^2}\,\cos u}$$

$\mathrm{D}=\dfrac{\partial}{\partial c}$; multiplying and dividing by ($-i$),

$$=\frac{(-)^{n+1}}{\pi}\frac{(n+m)!}{(n-m)!}\sqrt{\frac{\pi}{2\mathrm{D}}}\;\mathrm{I}_{n+\frac{1}{2}}(\mathrm{D}).\;(i)$$

$$\int_0^\infty d\lambda\int_0^\pi e^{i\lambda c-i\lambda\mu_1\zeta_1+\,i\lambda\sqrt{1-\mu_1^2}\,\sqrt{1-\zeta_1^2}\,\cos u}\cos mu\,du$$

since $\displaystyle\int_0^\infty e^{-kn}dn=\frac{1}{k}$,

$$=(-)^{n+1}\frac{(n+m)!}{(n-m)!}\sqrt{\frac{\pi}{2\mathrm{D}}}\;\mathrm{I}_{n+\frac{1}{2}}(\mathrm{D})\;i^{m+1}$$

$$\times\int_0^\infty e^{i\lambda c-i\lambda\mu_1\zeta_1}\,\mathrm{J}_m[\lambda\sqrt{1-\mu_1^2}\,\sqrt{1-\zeta^2}]\,d\lambda\qquad\ldots(\mathrm{u})$$

using the well-known formula, $e^{i\lambda\rho\cos u}=\mathrm{J}_0(\lambda\rho)+2\sum_{s=1}^\infty i^s\mathrm{J}_s(\lambda\rho)\cos su$.

Again, from above

$\mathrm{P}_n^m(\mu)\mathrm{Q}_n^m(\zeta)$

$$=\frac{1}{\pi}\frac{(n+m)!}{(n-m)!}(-)^n\sqrt{\frac{\pi}{2\mathrm{D}}}\;\mathrm{I}_{\lambda+\frac{1}{2}}(\mathrm{D})$$

$$\times\int_0^\pi\frac{\cos mu\,du}{c-\mu_1\zeta_1+\sqrt{1-\mu_1^2}\,\sqrt{1-\zeta_1^2}\,\cos u}$$

$$=\frac{1}{\pi}\frac{(n+m)!}{(n-m)!}(-)^n\sqrt{\frac{\pi}{2\mathrm{D}}}\;\mathrm{I}_{n+\frac{1}{2}}(\mathrm{D})\sum_0^\infty(2r+1)\;\mathrm{Q}_r\quad(c)$$

$$\times \int_0^\pi P_r\,[\mu_1\zeta_1 - \sqrt{1-\mu_1^2}\,\sqrt{1-\zeta_1^2}\cos u]\cos mu\,du$$

$$= (-)^{m+n}\frac{(n+m)\,!}{(n-m)\,!}\sqrt{\frac{\pi}{2D}}\;I_{n+\frac{1}{2}}(D)$$

$$\sum_{r=m}^{\infty}(2r+1)\frac{(r-m)\,!}{(r+m)\,!}\,Q_r(c)P_n^m(\mu_1)P_n^m(\zeta_1)$$

$$= \frac{(n+m)\,!}{(n-m)\,!}(-)^{n+1}\sqrt{\frac{\pi}{2D}}\;I_{n+\frac{1}{2}}(D)\;(-)^{m+1}\sqrt{\frac{\pi}{2}}$$

$$\sum_{r=m}^{\infty} i^{-r-1}\,P_r^m(\mu_1)\,P_r^m(\zeta_1)\int_0^\infty e^{i\lambda c}\,J_{r+\frac{1}{2}}(\lambda)\frac{d\lambda}{\sqrt{\lambda}}$$

$$\times\,(2r+1)\frac{(r-m)\,!}{(r+m)\,!} \qquad \ldots (\mathrm{v})$$

introducing the well-known integral formula for $Q_r(c)$.
Whence from (u) and (v), we may have a relation,

$$e^{-i\lambda\mu_1\zeta_1}\,J_m\,[\lambda\sqrt{1-\mu_1^2}\,\sqrt{1-\zeta_1^2}\,] = \sqrt{\frac{\pi}{2\lambda}}\sum_{r=m}^{\infty} i^{m-r}(2r+1)\frac{(r-m)\,!}{(r+m)\,!}$$

$$J_{r+\frac{1}{2}}(\lambda)P_r^m(\mu_1)P_r^m(\zeta_1)$$

or putting $\lambda=-kR$, $\mu_1=\cos\theta$, $(\zeta_1)=\cos\alpha$, we have the formula,

$$e^{ik\,R\cos\cos\alpha}\,J_m\,[k\,R\sin\theta\sin\alpha]$$

$$= \sqrt{\frac{\pi}{2kR}}\sum_{r=m}^{\infty} i^{r-m}(2r+1)\frac{(r-m)\,!}{(r+m)\,!}\,J_{r+\frac{1}{2}}(kR)$$

$$P_r^m(\cos\theta)\;P_r^m(\cos\alpha) \qquad \ldots (\mathrm{w})$$

which, when $m=0$ reduces to Bauer's formula.

Equations to determine the motion.

The velocity-potential of the motion should satisfy the following conditions:—

(i) $\nabla^2\phi=0.$

(ii) $\phi=0$, at infinity.

(iii) $\dfrac{\partial\phi}{\partial\zeta}=-v_1 b P_1(\mu)$, over the spheroid, b, $\zeta=\zeta_0$.

(iv) $\dfrac{\partial\phi}{\partial\zeta_0}=-v_2 a P_1(\mu_1)$, ... a, $\zeta_1={}_1\zeta_0$.

Assume

$$\phi=\sum_{n=0}^{\infty}\left[b_n P_n(\mu)q_n(\zeta)+a_n p_n(\mu_1)q_n(\zeta_1)\right] \qquad \text{... (6)}$$

Near the surface of the spheroid b, we have, by (2)

$$\phi=\sum_{r=0}^{\infty} b_r P_r(\mu)q_r(\zeta)+\sum_{n=0}^{\infty} a_n \sum_{r=0}^{\infty}(2r+1)\,{}_1\omega_n^0(r, c, a, b)\, P_r(\mu)p_r(\zeta).$$

Therefore the boundary condition over it gives,

$$-bv_1P_1(\mu)=\sum_{r=0}^{\infty} b_r P_r(\mu)q'_r(\zeta_0)$$

$$+\sum_{n=0}^{\infty}a_n\sum_{r=0}^{\infty}(2r+1)\,{}_1\omega_n^0(r, c, a, b)P_r(\mu)p'_r(\zeta_0).$$

whence, equating the co-efficients of the zonal Harmonics, we have,

$0=b.$

$$\left.\begin{aligned}-\frac{bv_1}{q'_1(\zeta_0)}&=b_1+3\frac{p'_1(\zeta_0)}{q'_1(\zeta_0)}\sum_{n=0}^{\infty}a_n\,{}_1\omega_n^0(1, c, a, b)\\ 0&=b_\nu+(2r+1)\frac{p'_r(\zeta_0)}{q'_r(\zeta_0)}\sum_{n=0}^{\infty}a_n\,{}_1\omega_n^0(r, c, a, b)\,;\ r>1\end{aligned}\right\} \qquad \text{... (7)}$$

Similarly the boundary condition over the other spheroid yields, on using (4).

$$\left.\begin{aligned}0&=a_0\\ -\frac{av_2}{q'_1(\zeta'_0)}&=a_1+3\,\frac{p'_1(\zeta'_0)}{q'_1(\zeta'_0)}\sum_{n=0}^{\infty} b_n\,{}_2\omega_n^0(1,c,a,b)\\ 0&=a_r+(2r+1)\,\frac{p'_r(\zeta'_0)}{q'_r(\zeta'_0)}\sum_{n=0}^{\infty} b_n\,{}_2\omega_n^0(r,c,a,b)\ ;\ r>1\end{aligned}\right\}\quad\ldots(8)$$

Two parallel co-axial circular discs.

In the case of two circular discs, we have, $\zeta_0={}_1\zeta_0=0$, and since

$$\left(\frac{d}{d\zeta}p_{2n}\right)_0=0,\ \frac{p'_{2n+1}(0)}{q'_{2n+1}(0)}=-\frac{2}{\pi}\ ,\ \text{and}\ \frac{1}{q'_1(0)}=-\frac{2}{\pi},$$

we see that the even co-efficients a_{2n}, b_{2n} vanish in the expression for the velocity-potential, and the above conditions reduce to

$$\left.\begin{aligned}b_1&=\frac{2b}{\pi}v_1+3.\ \frac{2}{\pi}\sum_{n=0}^{\infty} a_{2n+1}\,{}_1\omega_{2n+1}^0(1,c,a,b)\\ b_{2r+1}&=\frac{2}{\pi}(4r+3)\sum_{n=0}^{\infty} a_{2n+1}\,{}_1\omega_{2n+1}^0(2r+1,c,a,b),\ r>0\end{aligned}\right\}\ldots(9)$$

and

$$\left.\begin{aligned}a_1&=\frac{2av_2}{\pi}\ +3.\ \frac{2}{\pi}\sum_{n=0}^{\infty} b_{2n+1}\,{}_2\omega^0{}_{2n+1}(1,c,a,b)\\ a_{2r+1}&=\frac{2}{\pi}(4r+3)\sum_{n=0}^{\infty} b_{2n+1}\,{}_2\omega^0{}_{2n+1}(2r+1,c,a,b),\ r>0\end{aligned}\right\}\ldots(10)$$

The velocity-potential.

The velocity-potential is,

$$\phi=\sum_{n=0}^{\infty}\Big[b_{2n+1}P_{2n+1}(\mu)q_{2n+1}(\zeta)+a_{2n+1}P_{2n+1}(\mu_1)q_{2n+1}(\zeta_1)\Big]$$

putting $m=0$ by (2) $=\sqrt{\dfrac{\pi a}{2}}\displaystyle\int_0^{\infty} e^{-\lambda z_1}J_0(\lambda\rho)\frac{d\lambda}{\sqrt{\lambda}}\sum_0^{\infty} a_{2n+1}\,J_{2n+\frac{3}{2}}(\lambda a)$

$$+\sqrt{\frac{\pi b}{2}}\int_0^\infty e^{-\lambda z} J_0(\lambda\rho)\frac{d\lambda}{\sqrt{\lambda}}\sum_0^\infty b_{2n+1}J_{2n+\frac{3}{2}}(\lambda a)$$

$$=\sqrt{\frac{\pi}{2}}\int_0^\infty e^{-\lambda z} J_0(\lambda\rho)\frac{d\lambda}{\sqrt{\lambda}}\left[\sqrt{\frac{a}{b}}\, f(\lambda a)e^{-\lambda(z^1-z)} + \sqrt{\frac{b}{a}}\, F(\lambda b)\right] \quad \ldots (11)$$

where

$$f(\lambda a)=\sqrt{b}\sum_0^\infty a_{2n+1}J_{2n+\frac{3}{2}}(\lambda a)\ ;\ F(\lambda b)=\sqrt{a}\sum_0^\infty b_{2n+1}J_{2n+\frac{3}{2}}(\lambda b) \ldots (12)$$

Two Simultaneous Integral Equations.

Since

$$_1\omega^0{}_{2n+1}(2r+1,c,a,b)=\frac{\pi}{2}\sqrt{\frac{a}{b}}\int_0^\infty e^{-cx}J_{2n+\frac{3}{2}}(ax)\, J_{2r+\frac{3}{2}}(bx)\frac{dx}{x}, \text{from } (4b)$$

and

$$_2\omega^0{}_{2n+1}(2r+1, c, a, b)$$

$$=\frac{\pi}{2}\sqrt{\frac{b}{a}}\int_0^\infty e^{-cx}J_{2r+\frac{3}{2}}(ax)\, J_{2n+\frac{3}{2}}(bx)\frac{dx}{x}, \text{ from } (5b)$$

We have, from (10),

$$a_{2r+1}=\frac{2}{\pi}(4r+3)\sum_{n=0}^\infty b_{2n+1}\ {}_2\omega^0{}_{2n+1}(2r+1, c, a, b).$$

$$=(4r+3)\sqrt{\frac{b}{a}}\sum_{n=0}^\infty\int_0^\infty e^{-cx}J_{2r+\frac{3}{2}}(ax)\, J_{2n+\frac{3}{2}}(bx)\, b_{2n+1}\frac{dx}{x}.$$

by (5b),

$$=\sqrt{\frac{b}{a}}\int_0^\infty e^{-cx}\frac{dx}{x}(4r+3)J_{2r+\frac{3}{2}}(ax)\sum_0^\infty b_{2n+1}J_{2n+\frac{3}{2}}(bx)$$

$$=\sqrt{\frac{b}{a}}\int_0^\infty e^{-cx}\frac{dx}{x\sqrt{a}}F(bx)(4r+3)J_{2r+\frac{3}{2}}(ax) \quad \ldots (13)$$

Similarly, by (9), b_{2r+1}

$$= \sqrt{\frac{a}{b}} \int_0^\infty e^{-cx} \frac{dx}{x\sqrt{b}} f(ax)(4r+3) J_{2r+\frac{3}{2}}(bx), \qquad \ldots (14)$$

except when $r=0$, in which case, we must and $\frac{2av_2}{\pi}$ and $\frac{2bv_1}{\pi}$ on the right-hand sides respectively.

Let us now introduce another variable y, independent of x, and multiply the above equations (13) and (14) by $J_{2r+\frac{3}{2}}(ay)$ and $J_{2r+\frac{3}{2}}(by)$ respectively; then summing for all positive integral values of r, we have,

$$\sum_0^\infty a_{2r+1} J_{2r+\frac{3}{2}}(ay) = \frac{2av_2}{\pi} J_{\frac{3}{2}}(ay)$$

$$+ \sqrt{\frac{b}{a^2}} \int_0^\infty e^{-cx} \frac{dx}{x} F(bx) \sum_0^\infty (4r+3) J_{2r+\frac{3}{2}}(ax) J_{2r+\frac{3}{2}}(ay) \ldots (15)$$

and

$$\sum_0^\infty b_{2r+1} J_{2r+\frac{3}{2}}(by) = \frac{2bv_1}{\pi} J_{\frac{3}{2}}(by)$$

$$+ \sqrt{\frac{a}{b^2}} \int_0^\infty e^{-cx} \frac{dx}{x} f(ax) \sum_0^\infty (4r+3) J_{2r+\frac{3}{2}}(bx) J_{2r+\frac{3}{2}}(by) \ldots (16)$$

But we know from a special case of Gegenbauer's theorem,*

$$\sum_0^\infty (-)^s (2s+1) J_{s+\frac{1}{2}}(x) J_{s+\frac{1}{2}}(y) = \frac{2}{\pi} \sqrt{xy} \frac{\sin (x+y)}{x+y}, \quad x \neq y$$

and also

$$\sum_0^\infty (2s+1) J_{s+\frac{1}{2}}(x) J_{s+\frac{1}{2}}(y) = \frac{2}{\pi} \sqrt{xy} \frac{\sin (x-y)}{x-y},$$

* *Math. Ann.*, 11, 1871; G. N. Watson, The Theory of Bessel Functions, p. 525.

whence by subtraction,

$$\sum_{0}^{\infty}(4r+3)\,J_{2r+\frac{3}{2}}(x)\,J_{2r+\frac{3}{2}}(y)=\frac{1}{\pi}\sqrt{xy}\left[\frac{\sin(x-y)}{x-y}-\frac{\sin(x+y)}{x+y}\right] \quad \ldots (17)$$

Hence, from (15)

$$\frac{f(ay)}{\sqrt{by}}-\frac{2av_2}{\pi}\frac{J_{\frac{3}{2}}(ay)}{\sqrt{y}}$$

$$=\sqrt{\frac{b}{a^3}}\frac{1}{\pi}\int_0^{\infty}e^{-cx}\frac{dx}{\sqrt{x}}F(bx)\left[\frac{\sin a(x-y)}{x-y}-\frac{\sin a(x+y)}{x+y}\right] \ldots (18)$$

and from (16)

$$\frac{F(by)}{\sqrt{ay}}-\frac{2bv_1}{\pi}\frac{J_{\frac{3}{2}}(by)}{\sqrt{y}}$$

$$=\sqrt{\frac{a}{b^3}}\frac{1}{\pi}\int_0^{\infty}e^{-cx}\frac{dx}{\sqrt{x}}f(ax)\left[\frac{\sin b(x-y)}{x-y}-\frac{\sin b(x+y)}{x+y}\right] \ldots (19)$$

which are the two simultaneous Integral equations.

Two equal coaxial parallel circular discs.

In the case of two equal coaxial parallel circular discs, $a=b$, and the above two Integral equations reduce to

$$\left.\begin{aligned}
&\frac{f(ay)}{\sqrt{ay}}-\frac{2av_2}{\pi}\frac{J_{\frac{3}{2}}(ay)}{\sqrt{y}}\\
&=\frac{1}{\pi}\int_0^{\infty}e^{-cx}\frac{dx}{\sqrt{ax}}F(ax)\left[\frac{\sin a(x-y)}{x-y}-\frac{\sin a(x+y)}{x+y}\right]\\
\text{and}\quad &\frac{F(ay)}{\sqrt{ay}}-\frac{2av_1}{\pi}\frac{J_{\frac{3}{2}}(ay)}{\sqrt{y}}\\
&=\frac{1}{\pi}\int_0^{\infty}e^{-cx}\frac{dx}{\sqrt{ax}}f(ax)\left[\frac{\sin a(x-y)}{x-y}-\frac{\sin a(x+y)}{x+y}\right]
\end{aligned}\right\} \ldots \quad (20)$$

Two special cases.

(*i*) If $v_1=v_2$, we have by subtraction, from (20)

$$\frac{f(ay)}{\sqrt{ay}}-\frac{\mathrm{F}(ay)}{\sqrt{ay}}$$

$$=-\frac{1}{\pi}\int_0^\infty e^{-cx}\,dx\left[\frac{f(ax)}{\sqrt{ax}}-\frac{\mathrm{F}(ax)}{\sqrt{ax}}\right]\left[\frac{\sin a(x-y)}{x-y}-\frac{\sin a(x+y)}{x+y}\right]\ldots(21)$$

This is a homogeneous Integral equation, of which the only continuous solution is zero,* so that we have $f(ay)=\mathrm{F}(ay)$, and consequently,

$$\frac{f(ay)}{\sqrt{ay}}-\frac{2av_2}{\pi}\,\frac{\mathrm{J}_{\frac{3}{2}}(ay)}{\sqrt{y}}$$

$$=\frac{1}{\pi}\int_0^\infty e^{-cx}\,\frac{dx}{\sqrt{ax}}f(ax)\left[\frac{\sin a(x-y)}{x-y}-\frac{\sin a(x+y)}{x+y}\right]\qquad\ldots(\alpha)$$

(*ii*) If $v_1=-v_2$, we have by addition and arguing as before $f=-\mathrm{F}$, whence

$$\frac{f(ay)}{\sqrt{ay}}-\frac{2av_2}{\pi}\,\frac{\mathrm{J}_{\frac{3}{2}}(ay)}{\sqrt{y}}$$

$$=-\frac{1}{\pi}\int_0^\infty e^{-cx}\,\frac{dx}{\sqrt{ax}}f(ax)\left[\frac{\sin a(x-y)}{x-y}-\frac{\sin a(x+y)}{x+y}\right]\ldots\qquad(\beta)$$

Approximate solution of the Integral equations (α) *and* (β).

The same method of approximation may be applied both to (α) and (β). Let us consider the Integal Equation (β).

Putting $ay=z$, the equation (β) reduces to

* Whittaker and Watson, Mod. Analysis, 3rd ed., p. 217.

From (9) and (10) it may be easily seen in the case of equal circles, that $a_{2n+1}=b_{2n+1}$ and consequently from (12*a*) we have $f=\mathrm{F}$, which corroborates the validity of the conclusion and indicates that $-\frac{1}{\pi}$ is not a root of the equation $\mathrm{D}(\lambda)=0$, of the homogeneous integral equation.

$$\frac{f(z)}{\sqrt{z}}-\frac{2a^{\frac{3}{2}}v_2}{\pi}\frac{J_{\frac{3}{2}}(z)}{\sqrt{z}}=-\frac{1}{\pi}\int_0^\infty e^{-\frac{cx}{a}}\frac{dx}{\sqrt{x}}f(x)\left[\frac{\sin(x-z)}{x-z}-\frac{\sin(x+z)}{x+z}\right].$$

Let
$$I=\int_0^\infty e^{-kx}\left[\frac{\sin(x-z)}{x-z}-\frac{\sin(x+z)}{x+z}\right]dx,\ k=c/a.$$

$$=\int_0^\infty e^{-kx}\,dx\int_0^1 d\alpha\left[\cos\alpha(x-z)-\cos\alpha(x+z)\right]$$

$$=2\int_0^1\sin\alpha z\,d\alpha\int_0^\infty e^{-kx}\sin\alpha x\,dx,$$

by obviously valid inversion of order,

$$=2\int_0^1\frac{\sin\alpha z.\alpha d\alpha}{\alpha^2+k^2}=2\sum_{r=0}^\infty(-)^r\frac{z^{2r+1}}{(2r+1)!}\int_0^1\frac{\alpha^{2r+2}}{\alpha^2+k^2}d\alpha.$$

Therefore,

$$D^nI=(-)^n\int_0^\infty x^ne^{-kx}\left[\frac{\sin(x-z)}{x-z}-\frac{\sin(x+z)}{x+z}\right]dx$$

$$D=\frac{\partial}{\partial k}=a\frac{\partial}{\partial c}$$

$$=2\sum_{r=0}^\infty(-)^r\frac{z^{2r+1}}{(2r+1)!}\cdot D^n\int_0^1\frac{\alpha^{2r+2}}{\alpha^2+k^2}d\alpha \qquad \ldots(22)$$

Now let

$$\frac{f(z)}{\sqrt{z}}=\sum_0^\infty B^nz^n,$$

then the above integral equation becomes, by using (22)

$$\sum_{r=0}^\infty B_rz^r-\left(\frac{2a}{\pi}\right)^{\frac{3}{2}}v_2\sum_{r=0}^\infty(-)^r\frac{2r+2}{(2r+3)!}z^{2r+1}$$

$$=-\frac{2}{\pi}\sum_{r=0}^\infty B_n(-)^n\sum_{r=0}^\infty(-)^r\frac{z^{2r+1}}{(2r+1)!}D^n\int_0^1\frac{\alpha^{2r+2}d\alpha}{\alpha^2+k^2}.$$

from which we observe that only odd powers of z occur in $\frac{f(z)}{\sqrt{z}}$ and accordingly we have

$$B_{2r+1}-\left(\frac{2a}{\pi}\right)^{\frac{3}{2}} v_2 \,(-)^r\, \frac{2r+2}{(2r+3)!}$$

$$=\frac{2}{\pi}\frac{(-)^r}{(2r+1)!}\sum_{n=0}^{\infty} B_{2n+1} D^{2n+1}\int_0^1 \frac{a^{2r+2}da}{a^2+k^2}$$

$$=(-)^{r+1}\frac{2}{\pi}\cdot\frac{1}{(2r+1)!}\sum_{n=0}^{\infty} B_{2n+1}\left[\frac{(2n+2)!}{(2r+3)k^{2n+3}}\right.$$

$$\left.-\frac{(2n+4)!}{3!(2r+5)k^{2n+5}}+\frac{(2n+6)!}{5!(2r+7)k^{2n+7}}\cdots\right] \qquad \ldots (23)$$

Approximation.

(*i*) Central distance c being such that only the third power of a/c is retained, we have,

$$B_{2r+1}-\left(\frac{2a}{\pi}\right)^{\frac{3}{2}} v_2 \,(-)^r\, \frac{2r+2}{(2r+3)!}=(-)^{r+1}\frac{4}{\pi}\cdot\frac{B_1}{(2r+1)!}\cdot\frac{1}{2r+3}\left(\frac{a}{c}\right)^3.$$

Putting $r=0$, we have,

$$B_1=\left(\frac{2a}{\pi}\right)^{\frac{3}{2}}\frac{v_2}{3}\left[1-\frac{4}{3\pi}\left(\frac{a}{c}\right)^3\right];$$

and consequently

$$B_{2r+1}=(-)^r\left(\frac{2a}{\pi}\right)^{\frac{3}{2}} v_2\cdot\frac{2r+2}{(2r+3)!}\left[1-\frac{4}{3\pi}\left(\frac{a}{c}\right)^3\right].$$

(*ii*) If the fifth power of $\frac{a}{c}$ is also retained, then,

$$B_{2r+1}-(-)^r\left(\frac{2a}{\pi}\right)^{\frac{3}{2}} v_2\,\frac{(2r+2)}{(2r+3)!}=(-)^{r+1}\frac{2}{\pi}\cdot\frac{1}{(2r+1)!}$$

$$\left[\frac{1}{2r+3}\left(\frac{2B_1}{k^3}+\frac{4!B_3}{k^5}\right)-\frac{4B_1}{(2r+5)k^5}\right],$$

whence

$$B_1=\left(\frac{2a}{\pi}\right)^{\frac{3}{2}}\frac{v_2}{3}\left[1-\frac{4}{3\pi k^3}+\frac{16}{5\pi k^5}\right],\quad k=c/a.$$

$$B_3=-\left(\frac{2a}{\pi}\right)^{\frac{3}{2}}\frac{v_2}{30}\left[1-\frac{4}{3\pi k^3}+\frac{368}{105\pi k^5}\right],$$

and $$B_{2r+1}=(-)^r\left(\frac{2a}{\pi}\right)^{\frac{3}{2}}v_2\frac{2r+2}{(2r+3)!}\left[1-\frac{4}{3\pi}\left(\frac{1}{k^3}-\frac{4}{5k^5}\cdot\frac{8r+15}{2r+5}\right)\right].$$

The velocity-potential.

The velocity-potential of the motion is, from (12) and (13),

$$\phi=\sqrt{\frac{\pi}{2}}\int_0^\infty e^{-\lambda z}J_0(\lambda\rho)\frac{d\lambda}{\sqrt{\lambda}}f(\lambda a)\left[e^{-\lambda(z_1-z)}-1\right] \qquad (24)$$

where

$$f(x)=\sqrt{a}\sum_0^\infty a_{2n+1}J_{2n+\frac{3}{2}}(x) \text{ has been found above}$$

and $$a_{2r+1}=-\int_0^\infty e^{-xc}\frac{dx}{x\sqrt{a}}f(xa)\,(4r+3)\,J_{2r+\frac{3}{2}}(xa),$$

except when $r=0$, in which case, $\dfrac{2av_2}{\pi}$ must be added to the right hand side.

SECTION II.

STEADY ROTATION OF TWO CIRCULAR DISCS IN A VISCOUS LIQUID.

Part I.

Two equal circles.

We consider, as in the preceding section, two equal spheroids (both oblate) rotating with equal angular velocity Ω, about the common axis of revolution ; (ζ, μ, ω) and (ζ_1, μ_1, ω) denote, as before, the oblate spheroidal coordinates of a point in space, (z, ρ, ω) and (z_1, ρ_1, ω) denote

the corresponding cylindrical coordinates, with respect to the centres of the two spheroids as origin; the z-axes being measured positively in opposite senses along the axis of revolution; a, the radius of confocality and c, the central distance. The spheroids being equal must have the same ζ, say ζ_0. Then with the usual idea, we have the geometrical relations,

$$z+z_1=c,$$

i.e.,

$$\mu\zeta+\mu_1\zeta_1=c/a,$$

and

$$\frac{\rho}{a}=\sqrt{1-\mu^2}\ \sqrt{\zeta^2+1}\ =\sqrt{1-\mu_1^2}\ \sqrt{\zeta_1^2+1} \qquad \dots (1)$$

Dr. G. B. Jeffery* has shown that, when squares and products of velocities can be neglected, v, the velocity in the direction ω, (the other components=0), satisfies the equation

$$\nabla^2(v \sin w)=0,$$

so that the determination of v ensures the solution of the problem.

There being no slipping over the boundary, the boundary condition on either of the spheroids is

$$v=a\Omega\sqrt{\zeta_0^2+1}\ \frac{\sqrt{1-\mu^2}}{\sqrt{1-\mu_1^2}}=a\Omega\sqrt{\zeta_0^2+1}\ \frac{P_1^1(\mu)}{P_1^1(\mu_1)}.$$

v, therefore, satisfies the following conditions :—

$$\nabla^2(v \sin \omega)=0, \qquad \dots (2)$$

$$v=0, \text{ at infinity}, \qquad \dots (3)$$

$$\left.\begin{array}{l} v=a\Omega\sqrt{\zeta_0^2+1}\ P_1^1(\mu), \text{ on one spheroid,} \\ v=a\Omega\sqrt{\zeta_0^2+1}\ P_1^1(\mu_1), \text{ on the other.} \end{array}\right\} \qquad \dots (4)$$

Formation of an Integral equation.

$$\text{Assume } v=\sum_{n=1}^{\infty} a_n\left[P_n^1(\mu)q_n^1(\zeta)+P_n^1(\mu_1)q_n^1(\zeta_1)\right] \qquad \dots (5)$$

which evidently satisfies the conditions (2) and (3).

* *Proc. Lond. Math. Soc.*, 1915, p. 327.

Near the surface of a spheroid, putting $m=1$, and $b=a$, we have, by (3), section 1,

$$v=\sum_{r=1}^{\infty} a_r \mathrm{P}_r^1(\mu)q_r^1(\zeta)+\sum_{n=1}^{\infty} a_n \sum_{r=1}^{\infty}(2r+1)\,{}_1w_n^1(r,\,c,\,a)\mathrm{P}_r^1(\mu)p_r^1(\zeta).$$

The boundary condition over it gives

$$a\sqrt{\zeta_0^2+1}\;\mathrm{P}_1^1(\mu)\Omega=\sum_{r=1}^{\infty} a_r \mathrm{P}_r^1(\mu)q_r^1(\zeta_0)$$

$$+\sum_{n=1}^{\infty} a_n \sum_{r=1}^{\infty}(2r+1)\,{}_1w_n^1\,(r,\,c,\,a)\mathrm{P}_r^1(\mu)p_r^1(\zeta_0)$$

whence equating the coefficients of the associated functions, we have

$$\left.\begin{aligned} a\Omega\sqrt{\zeta_0^2+1} &= a_1 q_1^1(\zeta_0)+3p_1^1(\zeta_0)\sum_{n=1}^{\infty} a_{n\,1}w_n^1(1,\,c,\,a) \\ 0 &= a_r q_r^1(\zeta_0)+(2r+1)p_r^1(\zeta_0)\sum_{n=1}^{\infty} a_{n\,1}w_n^1(r,c,a), r>1 \end{aligned}\right\}\quad\ldots(\mathrm{x})$$

Therefore $\quad a_r=-(2r+1)\dfrac{p_r^1(\zeta_0)}{q_r^1(\zeta)_0}\displaystyle\sum_{n=1}^{\infty} a_{n\,1}\omega_n^1(r,\,c,\,a)\,;\; r=1,\,2,\,3\,\ldots\;(\mathrm{y})$

When $r=1$, there must be an addition of $\dfrac{a\Omega\sqrt{\zeta_0^2+1}}{q_1^1(\zeta_0)}$ on the right hand side.

Two circular discs.

In the case of two circular discs, we have $\zeta_0=0$. Again, as before,*

$$\frac{p_r^1(\mathrm{o})}{q_r^1(\mathrm{o})}=\mathrm{o}\text{ or }-2/\pi,$$

according as r is even or odd respectively; and $q_1^1(o)=-\dfrac{\pi}{2}$.

Consequently we notice that the coefficients a_{2r} do not occur in v, and putting $r=2m+1$, we have, from (y)

* Dr. Nicholson, *Phil. Trans. Royal Soc.*, Vol. 204, 1924, p. 53.

$$a_{2m+1} = \frac{2}{\pi}(4m+3)\sum_{n=0}^{\infty} a_{2n+1}\ {}_1\omega^1_{2n+1}(2m+1, c, a),$$

using the integral expression for q_r,

$$= -\frac{4m+3}{(2m+1)(2m+2)}\int_0^{\infty}\sum_{n=0}^{\infty}(2n+1\cdot(2n+2)a_{2n+1}$$

$$\times J_{2m+\frac{3}{2}}(\lambda a)J_{2n+\frac{3}{2}}(\lambda a)\frac{d\lambda}{\lambda}.\ e^{-\lambda c}$$

by (4), Section I,

$$= -\frac{4m+3}{(2m+1)(2m+2)}\int_0^{\infty} e^{-\lambda c} J_{2m+\frac{3}{2}}(\lambda a)\frac{d\lambda}{\lambda}$$

$$\times\left\{\sum_{n=0}^{\infty}(2n+1)(2n+2)a_{2n+1}J_{2n+\frac{3}{2}}(\lambda a)\right\}$$

$$= -\frac{4m+3}{(2m+1)(2+2)}\int_0^{\infty} e^{-\frac{\lambda c}{a}} J_{2m+\frac{3}{2}}(\lambda)\, f(\lambda)\frac{d\lambda}{\lambda}, \qquad \dots (6)$$

$$\text{where } f(x) = \sum_{n=0}^{\infty}(2n+1)(2n+2)a_{2n+1}J_{2n+\frac{3}{2}}(x) \qquad \dots (7)$$

Introducing as before, a new variable y and changing λ into x, multiplying by $(2m+1)(2m+2)J_{2m+\frac{3}{2}}(y)$ and summing for all integral values of m, we have, as in the foreging problem,

$$f(y)+\frac{4a\Omega}{\pi}J_{\frac{3}{2}}(y)$$

$$= -\int_0^{\infty} e^{-\frac{cx}{a}} f(x)\frac{dx}{x}\sum_{m=0}^{\infty}(4m+3)J_{2m+\frac{3}{2}}(x)\ J_{2m+\frac{3}{2}}(y)$$

$$= -\frac{\sqrt{xy}}{\pi}\int_0^{\infty} e^{-\frac{cx}{a}} f(x)\frac{dx}{x}\left[\frac{\sin(x-y)}{x-y}-\frac{\sin(x+y)}{x+y}\right] \qquad \dots (8)$$

The solution.

This is the same Integral equation as that discussed in the previous section and requires no further analysis.

The value of v.

From (5) and (2, Section I),

$$v=\sum_{n=0}^{\infty} a_{2n+1}\left[\mathrm{P}^1_{2n+1}(\mu)\ q^1_{2n+1}(\zeta)+\mathrm{P}^1_{2n+1}(\mu_1)q^1_{2n+1}(\zeta_1)\right]$$

$$=-\sqrt{\frac{\pi a}{2}}\int_0^\infty(e^{-\lambda z}+e^{-\lambda z_1})\,\mathrm{J}_1(\lambda\rho)\frac{d\lambda}{\sqrt{\lambda}}\sum_0^\infty(2n+1)(2n+2)a_{2n+1}\mathrm{J}_{2n+\frac{3}{2}}(\lambda a)$$

$$=-\sqrt{\frac{\pi a}{2}}\int_0^\infty(e^{-\lambda z}+e^{-\lambda z_1})\mathrm{J}_1(\lambda\rho)f(\lambda a)\frac{d\lambda}{\sqrt{\lambda}},$$

where the form of f has been approximately found above.

Alternative method of determining the coefficients.

The equations (x) may be solved differently thus :—

The coefficients a_{2r} vanishing, as has been previously observed, we have to solve

$$\left.\begin{aligned}-\frac{2a\Omega}{\pi}&=a_1-3.\frac{2}{\pi}\cdot\sum_{n=0}^{\infty}a_{2n+1}\omega^1{}_{2n+1}(1,\ c,\ a)\\ 0&=a_{2r+1}-\frac{2}{\pi}(4r+3)\sum_{n=0}^{\infty}a_{2n+1}\omega^1{}_{2n+1}(2r+1,\ c,\ a)\ ;\ r>0\end{aligned}\right\}$$

We may obtain after some calculation, from (4, Section I),

$$\omega^1{}_{2n+1}(2r+1,\ c,a)$$

$$=-2^{2r+2n+2}\frac{(2n+1)(2n+2)(2n+1)!(2r+1)!}{(2r+1)(2r+2)(4r+3)!(4n+3)!}\left[(2r+2n+2)!\left(\frac{a}{c}\right)^{2r+2n+3}\right.$$

$$\left.-(2r+2n+4)!\left(\frac{a}{c}\right)^{2r+2n+5}\left\{\frac{1}{2(4r+5)}+\frac{1}{2(4n+5)}\right\}+\dots\right]\quad\dots(9)$$

whence

(*i*) neglecting terms of order $\left(\dfrac{\text{linear dimensions}}{\text{central distance}}\right)^3$ and higher,

$$\left.\begin{array}{l} a_1 = -\dfrac{2a\Omega}{\pi} \\ a_{2r+1} = 0, r > 0 \end{array}\right\} \text{the corresponding value of } v \text{ is}$$

$$; \quad v = -\frac{2a\Omega}{\pi}\left[P_1^1(\mu)q_1^1(\zeta) + P_1^1(\mu_1)q_1^1(\zeta_1)\right] \quad \ldots(10)$$

(*ii*) neglecting terms of order $\left(\dfrac{\text{linear dimensions}}{\text{central distance}}\right)^6$ and higher,

$$a_1 = -\frac{2a\Omega}{\pi}\left[1 - \frac{4}{3\pi}\left(\frac{a}{c}\right)^3\right]$$

$$a_{2r+1} = 0, \; r > 0\,;$$

(*iii*) neglecting terms of order $\left(\dfrac{\text{linear dimensions}}{\text{central distance}}\right)^7$ and higher,

$$a_1 = -\frac{2a\Omega}{\pi}\left[1 - \frac{4}{\pi}\left\{\frac{1}{3}\left(\frac{a}{c}\right)^3 - \frac{4}{5}\left(\frac{a}{c}\right)^5\right\}\right] - \frac{32a\Omega}{9\pi^3}\left(\frac{a}{c}\right)^6$$

$$a_3 = \frac{16a\Omega}{45\pi^2}\left(\frac{a}{c}\right)^5$$

$$a_{2n+1} = 0, \; r > 1.$$

Part II.

Two unequal circles.

We assume the preliminary statements made in Part I.

The geometrical relations (a, b denoting the radii of confocality) are assumed as

$$a\mu\zeta + b\mu_1\zeta_1 = c \quad \text{and} \quad a\sqrt{1-\mu^2}\,\sqrt{\zeta^2+1} = b\sqrt{1-\mu_1^2}\,\sqrt{\zeta_1^2+1}.$$

We may easily deduce, as before, the following transformations,

$$P_n^m(\mu_1)q_n^m(\zeta_1) = \sum_{r=m}^{\infty} (2r+1)\,{}_1\omega_n^m(r, c, a, b)\, P_r^m(\mu)p_r^m(\zeta) \quad \ldots (11)$$

where ${}_1\omega_n^m(r, c, a, b)$

$$=(-)^{n+m}\frac{b}{a}\cdot\frac{(r-m)\,!}{(r+m)\,!}\cdot\frac{(n+m)\,!}{(n-m)\,!}\sqrt{\frac{\pi}{2D}}\,J_{n+\frac{1}{2}}(D)\,q_r\Big(\frac{c}{a}\Big),$$

$$D=b\frac{\partial}{\partial c} \quad \ldots(12)$$

and $$P_n^m(\mu)q_n^m(\zeta)=\sum_{r=m}^{\infty}(2r+1)\,{}_2\omega_n^m(r,c,b,a)\,P_r^m(\mu_1)p_r^m(\zeta_1) \quad \ldots(13)$$

where ${}_2\omega_n^m(r,c,b,a)$

$$=(-)^{n+m}\frac{a}{b}\cdot\frac{(r-m)\,!}{(r+m)\,!}\cdot\frac{(n+m)\,!}{(n-m)\,!}\sqrt{\frac{\pi}{2D'}}\,J_{n+\frac{1}{2}}(D')q_r\left(\frac{c}{b}\right),$$

$$D'=a\frac{\partial}{\partial c} \quad \ldots(14)$$

Assume

$$v=\sum_{n=1}^{\infty}\Big[A_nP_n^1(\mu)q_n^1(\zeta)+B_nP_n^1(\mu_1)q_n^1(\zeta_1)\Big] \quad \ldots(15)$$

Near the surface of the spheroid, a, $(\zeta=\zeta_0)$, we have, by (11)

$$v=\sum_{r=1}^{\infty}A_rP_r^1(\mu)q_r^1(\zeta)+\sum_{n=1}^{\infty}B_n\sum_{r=1}^{\infty}(2r+1)\,{}_1\omega_n^1(r,c,a,b)P_r^1(\mu)p_r^1(\zeta_1),$$

whence the boundary condition over it gives, by equating the coefficients of the harmonics,

$$\left.\begin{aligned} a\Omega\sqrt{\zeta_0^2+1} &= A_1q_1^1(\zeta_0)+3p_1^1(\zeta_0)\sum_{n=1}^{\infty}B_n\,{}_1\omega_n^1(1,c,a,b),\\ 0 &= A_rq_r^1(\zeta_0)+(2r+1)p_r^1(\zeta_0)\sum_{n=1}^{\infty}B_n\,{}_1\omega_n^1(r,c,a,b),\ r>1 \end{aligned}\right\}\ldots(A)$$

The boundary condition over the other spheroid gives, $\zeta_1={}_1\zeta_0$. by using (13)

$$\left.\begin{aligned} b\Omega_1\sqrt{\zeta_0^2+1} &= B_1q_1^1({}_1\zeta_0)+3\sum_{n=1}^{\infty}A_n\,{}_2\omega_n^1(1,c,b,a)p_1^1({}_1\zeta_0),\\ 0 &= B_rq_0({}_1\zeta_0)+(2r+1)p_r^1({}_1\zeta_0)\sum_{n=1}^{\infty}A_n\,{}_2\omega_n^1(r,c,b,a),r>1 \end{aligned}\right\}\ldots(B)$$

the boundary condition on the boundaries, subject to no slipping, being v=velocity in the direction of ω

$$=\Omega a\sqrt{1-\mu^2}\ \sqrt{\zeta_0^2+1}=a\Omega\sqrt{\zeta_0^2+1}\ \mathrm{P}_1^1(\mu),\text{ over the spheroid } \zeta_0=\zeta,$$

$$\mathrm{V}=b\Omega_1\sqrt{{}_1\zeta_0^2+1}\mathrm{P}_1^1(\mu_1),\text{ over the spheroid } \zeta_1={}_1\zeta_0,\ \Omega \text{ and } \Omega_1$$

being their angular velocities.

Two unequal parallel circular discs.

If $\zeta_0={}_1\zeta_0=0$, the two spheroids reduce to two un-equal circular discs of radii a and b respectively; since

$$\frac{p'_{2n}(0)}{q'_{2n}(0)}=0,\ \frac{p'_{2n+1}(0)}{q'_{2n+1}(0)}=-2/\pi\ ;\ \frac{1}{q'_1(0)}=-\frac{2}{\pi},$$

the equations (A) and (B) become

$$\left.\begin{aligned}
-\frac{2a\Omega}{\pi}&=\mathrm{A}_1-\frac{2}{\pi}.3.\sum_{n=0}^{\infty}\mathrm{B}_{2n+1}\ {}_1\omega_{2n+1}^1(1,c,a,b)\\
0&=\mathrm{A}_{2r+1}-\frac{2}{\pi}(4r+3)\sum_{n=0}^{\infty}\mathrm{B}_{2n+1}\ {}_1\omega_{2n+1}^1(2r+1,c,a,b)\\
0&=\mathrm{A}_{2r}\ ;
\end{aligned}\right\}\cdots\ (\mathrm{A}')$$

and

$$\left.\begin{aligned}
-\frac{2b\Omega_1}{\pi}&=\mathrm{B}_1-\frac{2}{\pi}.3.\sum_{n=0}^{\infty}\mathrm{A}_{2n+1}\ {}_2\omega_{2n+1}^1(1,c,b,a)\\
0&=\mathrm{B}_{2r+1}-\frac{2}{\pi}(4r+3)\sum_{n=0}^{\infty}\mathrm{A}_{2n+1}\ {}_2\omega_{2n+1}^1(2r+1,c,b,a),r>0\\
0&=\mathrm{B}_{2r},
\end{aligned}\right\}\cdots(\mathrm{B})$$

whence, substituting from (B′) in (A′),

$$\left.\begin{aligned} A_1 - \sum_{n=0}^{\infty} A_{2n+1}\theta_{2n+1,1} &= -\frac{2a\Omega}{\pi} - \frac{2b\Omega_1}{\pi^2}\ {}_1\omega_1^1(1, c, a, b) \\ A_{2r+1} - \sum_{n=0}^{\infty} A_{2n+1}\, \theta_{2n+1, 2r+1} & \\ &= -\frac{4b\Omega_1}{\pi^2}\ (4r+3)_1\omega_1^1(2r+1, c, a, b), r>0 \end{aligned}\right\}$$

where

$$\theta_{2n+1,\ 2r+1}$$

$$= \frac{4}{\pi^2}(4r+3) \sum_{p=0}^{\infty} (4p+3)_1\omega_{2p+1}^1(2r+1, c, a, b,)_2\omega_{2n+1}^1(2p+1, c, b, a)$$

and, also

$$\left.\begin{aligned} B_1 - \sum_{n=0}^{\infty} B_{2n+1}\phi_{2n+1,} &= -\frac{2b\Omega_1}{\pi} - \frac{12a\Omega}{\pi^2}\ {}_2\omega_1^1(1, c, b, a) \\ B_{2n+1} - \phi_{2n+1, 2r+1} &= -\frac{4a\Omega}{\pi^2}\ (4r+3)_2\omega_1^1(2r+1, c, b, a)\ ;\ r>0 \end{aligned}\right\}$$

where

$$\phi_{2n+1, 2r+1}$$

$$= \frac{4}{\pi^2}(4r+3) \sum_{p=0}^{\infty} (4p+3)_1\omega_{2n+1}^1(2p+1, c, a, b)_2\omega_{2p+1}^1(2r+1, c, b, a)$$

Approximate determination of the coefficients.

We have from (12)

$${}_1\omega_{2n+1}^1(2r+1, c, a, b)$$

$$= -2^{2r+2n+2}\frac{(2n+1)(2n+2)}{(2r+1)(2r+2)}\left[\frac{(2r+1)!\ (2r+2n+2)!}{(4r+3)\ !}\ \frac{b^{2n+2}a^{2r+1}}{c^{2r+2n+3}}\right.$$

$$\times\frac{(2n+1)\ !}{(4n+3)\ !} - \frac{(2n+1)\ !}{(4n+3)\ !}\frac{(2r+1)\ !}{(4r+3)\ !}b^{2n+2}a^{2r+1}\ \frac{(2r+2n+4)\ !}{2c^{2r+2n+5}}\left\{\frac{1}{4n+5}\ b^2\right.$$

$$\left.\left. + \frac{1}{4r+5}\ a^2\right\} + \ldots\right]$$

and from) 14), $_2\omega^1_{2n+1}(2r+1,\ c,\ b,\ a)$ = a similar expression in which a and b are interchanged.

$$\theta_{2n+1,2r+1} = \frac{4}{\pi^2}(4r+3)\sum_{p=0}^{\infty}(4p+3)2^{2r+2n+4p+4}$$

$$\times\frac{(2p+2n+2)!\,(2p+2r+2)!\,(2n+1)(2n+2)(2n+1)!\,(2r+1)!\,\{(2p+1)!\}^2}{(2r+1)(2r+2)(4r+3)\,!\,(4n+3)\,!\,(4p+3)\,!^2}$$

$$\times\frac{a^{2r+2n+3}b^{4p+3}}{c^{2n+2r+4p+6}} + \ldots$$

$\phi_{2n+1,2r+1}$ = a similar expression in which a, b are interchanged + ...

and $$\theta_{1,1} = \frac{16}{9\pi^2}\,\frac{a^3b^3}{c^6} + \ldots$$

$$\phi_{1,1} = \frac{16}{9\pi^2}\,\frac{c^3b^3}{c^6} + \ldots$$

whence

(*i*) neglecting terms of order $\left(\dfrac{\text{linear dimensions}}{\text{central distance}}\right)^3$ and higher,

$$\left.\begin{array}{l} A_1 = -\dfrac{2a\Omega}{\pi} \\ A_{2r+1} = 0,\ r>0 \end{array}\right\} \quad \left.\begin{array}{l} B_1 = -\dfrac{2b\Omega_1}{\pi} \\ B_{2r+1} = 0,\ r>0 \end{array}\right\}.$$

The corresponding value of v is from (15)

$$v = -\frac{2}{\pi}\,[a\Omega\,P_1^1(\mu)q_1^1(\zeta) + b\Omega_1 P_1^1(\mu_1)q_1^1(\zeta_1)];$$

(*ii*) neglecting terms of order $\left(\dfrac{\text{linear dimensions}}{\text{central distance}}\right)^6$ and higher,

$$A_1 = -\frac{2a}{\pi}\left[\Omega - \frac{4\Omega_1}{3\pi}\left(\frac{b}{c}\right)^3\right]; \qquad B_1 = -\frac{2b}{\pi}\left[\Omega_1 - \frac{4\Omega}{3\pi}\left(\frac{a}{c}\right)^3\right];$$

$$A_{2r+1}=0,\ r>0\,; \qquad B_{2r+1}=0,\ r>0\,;$$

(*iii*) neglecting terms of order $\left(\dfrac{\text{linear dimensions}}{\text{central distance}}\right)^7$ and higher,

$$A_1=-\frac{2a\Omega}{\pi}\left[1+\frac{16}{9\pi^2}\frac{a^3b^3}{c^6}\right]+\frac{8a\Omega_1 b^3}{3\pi^2 c^3}\left(1-\frac{6}{5}\frac{a^2+b^2}{c^2}\right),$$

$$A_3=\frac{16}{15\pi}\cdot\frac{a^3b^2}{c^5},$$

$$A_{2r+1}=0,\ r>1.$$

$$B_1=-\frac{2b\Omega_1}{\pi}\left[1+\frac{16}{9\pi^2}\frac{a^3b^3}{c^6}\right]+\frac{8b\Omega}{3\pi^2}\frac{a^3}{c^3}\left(1-\frac{6}{5}\frac{a^2+b^2}{c^2}\right).$$

$$B_3=\frac{16}{15\pi}\cdot\frac{a^2b^3}{c^5}.$$

$$B_{2r+1}=0,\ r>1.$$

The corresponding value of v can be written down at once.

The couple necessary to maintain the rotation.

If (ξ, η, ω) be a system of orthogonal coordinates, the elements of arcs measured along the normals to the surfaces ξ, η, ω=const. are

$$\frac{d\xi}{h_1},\ \frac{dn}{h_2},\ \frac{d\omega}{h_3},$$

respectively; and if (u', v', w') denote the corresponding components of velocity, we have, with the notation of stress-components,

$$\widehat{\xi\omega}=\mu\left[\frac{h_3}{h_1}\frac{\partial}{\partial\omega}(h_1u')+\frac{h_1}{h_3}\frac{\partial}{\partial\xi}(h_3w')\right],$$

where μ=coefficient of viscosity. Moreover, if (ξ, η) be conjugate functions of (z, ρ), we have $h_1=h_2$ and $h_3=\dfrac{1}{\rho}$ and if the solid be defined by ξ=const., we have, for the tangential stress on its surface in the direction perpendicular to the axis, $\mu h_1\rho\dfrac{\partial}{\partial\xi}\left(\dfrac{v}{\rho}\right)$ and the total

couple exerted by the fluid on the solid is

$$G=2\pi\mu \int \rho^3 \frac{\partial}{\partial \xi}\left(\frac{v}{\rho}\right) d\eta,$$

the value of the integrand being taken on the surface of the solid and the integration extending round the contour of the solid

$$=2\pi\mu \int \rho^3 \frac{\partial}{\partial \zeta}\left(\frac{v}{\rho}\right) \frac{\partial \zeta}{\partial \xi} \frac{\partial \eta}{\partial \mu}\, d\mu, \qquad \ldots \text{(z)}$$

where (ζ, μ, ω) form an orthogonal system such that

$$\zeta=f_1(\xi) \text{ and } \mu=f_2(\eta).$$

In the case of an oblate spheroid we may take $\zeta=\sinh \xi$ and $\mu=\cos \eta$, so that the formula (z) takes the form

$$G=-2\pi\mu \int \rho^3 \frac{\partial}{\partial \zeta}\left(\frac{v}{\rho}\right) \sqrt{\frac{\zeta^2+1}{1-\mu^2}}\, d\mu.$$

Thus, taking v from (iii) of the preceding section, we have,

$$\frac{\partial}{\partial \zeta}\left(\frac{v}{\rho}\right)=\frac{A_1}{a}\left(\frac{\partial^2 q_1}{\partial \zeta^2}\right)+\frac{A_3}{a}\frac{d^2 q_3}{d\zeta^2}\cdot\frac{P_3^1(\mu)}{\sqrt{1-\mu^2}}$$

$$+\sum_{r=1}^{\infty}\frac{(2r+1)}{a}\frac{P_r^1(\mu)}{\sqrt{1-\mu^2}}\frac{d}{d\zeta}\left(\frac{p_r^1(\zeta)}{\sqrt{\zeta^2+1}}\right)\left[B_{11}\omega_1^1+B_{31}\omega_3^1\right]$$

and the couple on the circle a

$$G=-2\pi\mu a^2 A_1\left(\frac{\partial^2 q_1}{\partial \zeta^2}\right)_0 \int_{-1}^{1}(1-\mu^2)\,d\mu.$$

The other terms vanishing due to the well-known integral properties of the products of associated functions and also due to the fact that

$$\left\{\frac{d}{d\zeta}\left(\frac{p_1^1(\zeta)}{\sqrt{\zeta^2+1}}\right)\right\}_0=0$$

$$=-\frac{16\pi\mu}{3}a^2 A_1$$

$$=\frac{32\mu a^3}{3}\left[\Omega\left(1+\frac{16}{9\pi^2}\frac{a^3b^3}{c^6}\right)-\frac{4\Omega_1}{3\pi}\frac{b^3}{c^3}\left(1-\frac{6}{5}\frac{a^2+b^2}{c^2}\right)\right].$$

Observation.

(*a*) If terms of order $\left(\dfrac{\text{Linear dimension}}{\text{central distance}}\right)^3$ and higher be neglected, we have from above

$$G=\frac{32\mu a^3}{3}\Omega,$$

which is Dr. Jeffery's result for the rotation of a single circular disc. We observe that the couple on either of the discs is quite independent of the rotation of the other, as we should expect.

(*b*) If the circle b be constrained to rotate with a given spin Ω_1, the steady angular velocity with which the circle a would rotate if allowed to move freely, would be that for which the couple on it vanishes. Thus, corresponding to (*iii*) of the preceding section, the steady angular velocity

$$=\frac{\dfrac{4\Omega_1}{3\pi}\dfrac{b^3}{c^3}\left(1-\dfrac{6}{5}\dfrac{a^2+b^2}{c^2}\right)}{1+\dfrac{16}{9\pi^2}\dfrac{a^3b^3}{c^6}}$$

$$=\frac{4\Omega_1 b^3}{3\pi c^3}\left[1-\frac{6}{5}\frac{a^2+b^2}{c^2}\right],$$ to our order of approximation.

3

The Theory of Intermittent Action and Band Spectra in Infra-red

By

Dhirendra Kumar Sen and Baidyanath Biswas

The theory of intermittent action as developed by Kar has been successfully applied in several cases.* In a paper published in "Zeitschrift fur Physik" Kar and Biswas † found out an expression for the energy of an an-harmonic oscillator due to the asymmetric term x^2 in the differential equation of motion. In the present paper an attempt has been made to obtain the same expression from a different view of the intermittent theory.

1. According to this theory the radiation, interacting on the oscillator, imparts impulses of equal magnitudes, mu, where

$$\frac{1}{2}mu^2=h\nu \qquad \ldots (1)$$

If the oscillator receives the impulses when it is at its mean positions, its energy will be $n^2h\nu$ after n impulses. If, however, the impulses are given when the oscillator is at rest (momentarily) and is on the point of moving in the direction of the impulse, the energy of the oscillator will be $nh\nu$.

1st case. Let the velocity of the oscillator just before the nth impulse be denoted by u_{n-1}, then the velocity just after the impulse will be u_n = velocity before the next impulse.

$$\therefore mu=mu_n-mu_{n-1} \qquad \ldots (2)$$

* Kar—*Phys. Rev.*, Vol. 21, p. 695, 1923; *Phys. Zeit.*, 24, 63, 1923; *Zs. f. Physik.*, Bd. 51, 5-6 heft, 416, 1929.

Kar and Ghose—*Phys. Zeit.*, 29, 143, 1928.

Ghose—*Phys. Zeit.*, 30, 160, 1929.

Kar and Mazumdar—*Zs. f. Phys.*, Bd. 53, 34, 4 heft, 308, 1929.

Biswas—*Journal of the Indian Mathematical Society*, Vol. XVIII, No. 8, 1930.

† Kar and Biswas, Bd. 59, 570, 1930.

$$\therefore u_n = u_{n-1} + u = n_{n-2} + 2u, \text{ etc.} = nu$$

Energy, $E^1(n) = \frac{1}{2} m u_n^2 = n^2 h\nu$ from (1) ... (3)

2nd case. Let the oscillator be at distance $-B_{n-1}$ from its mean position when it receives the n^{th} impulse. Its energy before the impulse

or $$E(n-1) = \frac{1}{2} m\lambda^2 B_{n-1}^2 \quad \dots \quad (4)$$

where $\ddot{x} + \lambda^2 x = 0$ is its equation of motion.

Since the body was at rest just before, the impulse generates a velocity equal to u. The subsequent motion being harmonic, the energy after the impulse,

or $$E(n) = \frac{1}{2} m\lambda^2 B_{n-1}^2 + \frac{1}{2} m u^2 \quad \dots \quad (5)$$

$$= E(n-1) + h\nu \quad \dots \quad (6)$$

It follows that $E(n) = E(n-2) + 2h\nu$ and so on,

$$= E(1) + (n-1)h\nu \quad \dots \quad (7)$$

But $$E(1) = \frac{1}{2} m u^2 = h\nu$$

$$\therefore E(n) = nh\nu \quad \dots \quad (8)$$

2. We shall next consider the case of an-harmonic oscillator subject to the second hypothesis and shall try to obtain an expression for its energy corresponding to the result (8) for harmonic oscillator.

Let x, $\dot{x}$ denote the position and velocity of the oscillator up to certain approximation, and ξ, $\dot{\xi}$ the changes in the above quantities for an additional degree of approximation.

Change in potential energy $= \frac{1}{2} m\lambda^2 [(x+\xi)^2 - x^2]$

$$= m\lambda^2 \xi \left(x + \frac{\xi}{2} \right)$$

and similarly change in kinetic energy $= m\dot{x}(\dot{x} + \frac{1}{2}\dot{\xi})$.

As we have to deal with mean values of the expressions for the energy in the case of an-harmonic oscillator we make the following conventions:

The mean value of the change in potential energy, for further approximation, will be taken to be positive or negative according as the mean value of ξ (or $\bar{\xi}$) is $+ve$ or $-ve$.

Similarly for the change in the kinetic energy.

3. The equation of motion for the harmonic oscillator is

$$\ddot{x}+\lambda^2 x=0 \qquad \dots \quad (9)$$

and that for the an-harmonic is

$$\ddot{x}+\lambda^2 x=-ax^2-bx^3 \qquad \dots \quad (10)$$

where a, b are quantities of increasing orders of smallness.

Let $-b_{n-1}$ be the distance of the an-harmonic oscillator when it receives the n^{th} impulse. It is evident that b_{n-1} will differ from B_{n-1} only in terms containing a, b,...Both the oscillators possess the velocity u just after the impulse so that the initial conditions for them are

$$x=-b_{n-1} \text{ or } -B_{n-1} \text{ and } \dot{x}=u.$$

The solution of (9) satisfying the initial conditions is

$$x=-B_{n-1}\cos\lambda t+\frac{u}{\lambda}\sin\lambda t=x_0 \text{ (suppose)} \qquad \dots \quad (11)$$

It is clear that for subsequent motion

$$E(n)=\tfrac{1}{2}m\lambda^2 B_n^2.$$

From (5) we have

$$B_n^2=B_{n-1}^2+\frac{u^2}{\lambda^2}=B_{n-2}^2+\frac{2u^2}{\lambda^2}\text{, etc.} \qquad \dots \quad (12)$$

and ultimately

$$B_n^2=\frac{nu^2}{\lambda^2} \qquad \dots \quad (12a)$$

4. Retaining only the term in a in equation (10) and substituting (11) on the right-hand side we get the following equation as the equation of motion of the an-harmonic oscillator,

$$\ddot{x}+\lambda^2 x=-a\left(-b_{n-1}\cos\lambda t+\frac{u}{\lambda}\sin\lambda t\right)^2 \qquad \dots \quad (13)$$

b_{n-1} being writtien for B_{n-1} for the sake of better approximation. On reduction this becomes

$$\ddot{x}+\lambda^2 x=-\frac{a}{2}\left[a_1+a_2\cos 2\lambda t+a_3\sin 2\lambda t\right] \quad \ldots \quad (14)$$

where

$$a_1=b_{n-1}^2+\frac{u^2}{\lambda^2},\ a_2=b_{n-1}^2-\frac{u^2}{\lambda^2}\text{ and } a_3=-\frac{2ub_{n-1}}{\lambda} \quad \ldots \quad (15)$$

The solution of (14) is

$$x=A\cos\lambda t+B\sin\lambda t-\frac{a}{2\lambda^2}\left[a_1-\frac{1}{3}a_2\cos 2\lambda t-\frac{1}{3}a_3\sin 2\lambda t\right] \ldots (16)$$

The initial conditions $x=-b_{n-1}$, $\dot{x}=u$, give with (15)

$$-b_{n-1}=A-\frac{a}{3\lambda^2}\left(b_{n-1}^2+\frac{2u^2}{\lambda^2}\right)$$

and

$$u=B\lambda-\frac{2}{3}\frac{aub_{n-1}}{\lambda^2}.$$

Substituting in (16) we get

$$x=-b_{n-1}\cos\lambda t+\frac{u}{\lambda}\sin\lambda t-\frac{a}{2\lambda^2}\left[a_1-\frac{a_2}{3}\cos 2\lambda t-\frac{1}{3}a_3\sin 2\lambda t+\right.$$

$$\left.+a_4\cos\lambda t+a_5\sin\lambda t\right] \quad \ldots \quad (17)$$

where

$$a_4=-\frac{2}{3}\left(b^2{}_{n-1}+\frac{2u^2}{\lambda^2}\right)\text{ and } a_5=-\frac{4}{3}\frac{ub_{n-1}}{\lambda} \quad \ldots \quad (18)$$

and

$$\dot{x}=b_{n-1}\lambda\sin\lambda t+u\cos\lambda t-\frac{a}{2\lambda}\left[\frac{2}{3}a_2\sin 2\lambda t-\frac{2}{3}a_3\cos 2\lambda t\right.$$

$$\left.-a_4\sin\lambda t+a_5\cos\lambda t\right] \quad \ldots \quad (19)$$

5. Our object is to calculate b_n in terms of b_{n-1} etc. We put

$$R^2_{n-1}=b^2_{n-1}+\frac{u^2}{\lambda^2} \text{ and } \tan k=\frac{\lambda b_{n-1}}{u} \qquad \dots (20)$$

so that

$$x=R_{n-1}\sin(\lambda t-k)-\frac{a}{2\lambda^2}\left[a_1-\dots+a_5\sin\lambda t\right] \qquad \dots (21)$$

and

$$\dot{x}=\lambda R_{n-1}\cos(\lambda t-k)-\frac{a}{2\lambda}\left[\frac{2}{3}a_2\sin 2\lambda t-\dots+a_5\cos\lambda t\right] \qquad \dots (22)$$

As each of x and $\dot{x}$ consists of two parts one of which is small we may assume that $\dot{x}=0$ and x has the greatest negative value when $\lambda t-k=3\pi/2+\theta$ where θ is small (of order a), because $\lambda t-k=3\pi/2$ makes the first part of $\dot{x}=0$ and that of $x=-R_{n-1}$, its greatest negative value. The value of θ can be obtained by putting $\dot{x}=0$ and $\lambda t-k=3\pi/2+\theta$ in (22); that is, from

$$0=\lambda R_{n-1}\sin\theta-\frac{a}{2\lambda}\left[-\frac{2}{3}a_2\sin 2k+\frac{2}{3}a_3\cos 2k\right.$$

$$\left.+a_4\cos k+a_5\sin k\right] \qquad \dots (23)$$

θ being neglected in evaluating the expression within the brackets for first approximation. As θ is small we may put $\sin\theta=\theta$ and $\cos\theta=1$. The equation (21) now gives the value of x which is evidently $-b_n$ for this value of $\lambda t-k$. We obtain

$$-b_n=-R_{n-1}\cos\theta-\frac{a}{2\lambda^2}\left[a_1+\frac{1}{3}a_2\cos 2k+\frac{1}{3}a_3\sin 2k\right.$$

$$\left.+a_4\sin k-a_5\cos k\right] \qquad \dots (24)$$

Substituting from (15), (18) and (20) and simplifying we get

$$-b_n=-R_{n-1}\left[1+\frac{a}{3\lambda^2}R_{n-1}\left\{1-\left(\frac{b_{n-1}}{R_{n-1}}\right)^3\right\}\right] \qquad \dots (25)$$

$$\therefore \lambda^2 b^2_n=\lambda^2 R^2_{n-1}\left[1+\frac{2a}{3\lambda^2}R_{n-1}\left\{1-\left(\frac{b_{n-1}}{R_{n-1}}\right)^3\right\}\right]$$

up to first power of a,

$$=\lambda^2 b_{n-1}^2+u^2+\frac{2a}{3}\left(R_{n-1}^3-b_{n-1}^3\right) \quad \dots (26)$$

Up to this approximation we may put $b_{n-1}=B_{n-1}$ and therefore $R_{n-1}=B_n$ [from (20)] within the brackets ; thus

$$\lambda^2 b_n^2=\lambda^2 b_{n-1}^2+u^2+\frac{2a}{3}(B_n^3-B_{n-1}^3) \quad \dots (27)$$

Similarly

$$\lambda^2 b_{n-1}^2=\lambda^2 b_{n-2}^2+u^2+\frac{2a}{3}(B_{n-1}^3-B_{n-2}^3),$$

and so on. Lastly

$$\lambda^2 b_1^2=u^2+\frac{2a}{3}(B_1^3).$$

Hence adding and removing common terms we have

$$\lambda^2 b_n^2=nu^2+\frac{2a}{3}B_n^3=\lambda^2 B_n^2+\frac{2a}{3}B_n^3 \quad \dots (28)$$

6. *Next approximation.* Our procedure will be similar to that employed in the previous article. θ will still be given by (23) as we shall require the values of $\cos\theta$ or $1-\frac{1}{2}\theta^2$, $a\cos\theta$ and $a\sin\theta$. Substituting the values of $\cos k$ and $\sin k$ and simplifying, we get

$$\theta=-\frac{au}{3\lambda^3}\left[2+\left(\frac{b_{n-1}}{R_{n-1}}\right)^2\right] \quad \dots (29)$$

And from (21) and (24) we get the additional part of $-b_n$ for this approximation to be

$$=\tfrac{1}{2}R_{n-1}\theta^2-\frac{a\theta}{2\lambda^2}\left[-\tfrac{2}{3}a_2\sin 2k+\tfrac{2}{3}a_3\cos 2k+a_4\cos k+a_5\sin k\right] \dots (30)$$

by putting $\lambda t=\frac{3\pi}{2}+(k+\theta)$ within the brackets as is necessary upto

this stage. On simplification this reduces to $-\frac{1}{2}R_{n-1}\theta^2$ ... (31)

$$\therefore -b_n = -R_{n-1}\left[1+\frac{a}{3\lambda^2}R_{n-1}\left\{1-\left(\frac{b_{n-1}}{R_{n-1}}\right)^3\right\}+\tfrac{1}{2}\theta^2\right] \quad \cdots(32)$$

from (25) and (31).

$$\therefore \lambda^2 b_n^2 = \lambda^2 R_{n-1}^2\left[1+\frac{2a}{3\lambda^2}R_{n-1}\left\{1-\left(\frac{b_{n-1}}{R_{n-1}}\right)^3\right\}\right.$$

$$\left.+\theta^2+\frac{a^2}{9\lambda^4}R_{n-1}^2\left\{1-\left(\frac{b_{n-1}}{R_{n-1}}\right)^3\right\}^2\right]$$

$$=\lambda^2 R_{n-1}^2 + \frac{2a}{3}\left\{R_{n-1}^3 - b_{n-1}^3\right\}$$

$$+\frac{a^2 R_{n-1}^2}{9\lambda^4}\left[R_{n-1}^2\left\{1-\left(\frac{b_{n-1}}{R_{n-1}}\right)^3\right\}^2\right.$$

$$\left.+\frac{u^2}{\lambda^2}\left\{2+\left(\frac{b_{n-1}}{R_{n-1}}\right)^2\right\}^2\right]$$

With the help of (20) and (28) and the method of the last article we can obtain $\lambda^2 b_n^2$ in the following form:

$$\lambda^2 b_n^2 = \lambda^2 B_n^2 + \frac{2a}{3}B_n^3 + a^2 f(n), \text{ say.} \quad \cdots (33)$$

7. Now (17) and (18) can be put as

$$x = x_0 + \xi \text{ and } \dot{x} = \dot{x}_0 + \dot{\xi}$$

where x_0 is free from a, *i.e.*, x_0 relates to harmonic oscillator and is given by (11). Energy, E_0, due to x_0 is given by (8).

Potential energy, or P, due to $\xi = m\lambda^2\xi\left(x_0+\frac{\xi}{2}\right)$

$$=-\frac{ma}{2}\left[a_1-\frac{1}{3}a_2\cos 2\lambda t-\frac{1}{3}a_3\sin 2\lambda t+a_4\cos\lambda t+a_5^1\sin\lambda t\right]$$

$$\times\left[x_0-\frac{a}{4\lambda^2}\left\{a_1-\ldots+a_5 \sin \lambda t\right\}\right]$$

$$+\text{ Part due from } \frac{1}{2}m\lambda^2\left(-b_{n-1}\cos \lambda t+\frac{u}{\lambda}\sin \lambda t\right)^2 \ldots \quad (34)$$

as b_{n-1} contains a [See (28) and (23)].

$$\therefore \bar{P}=-\frac{ma}{2}\left[-\frac{a_4 B_{n-1}}{2}+\frac{a_5 u}{2\lambda}\right]+\frac{ma^2}{8\lambda^2}\left\{a_1^2+\frac{a_2^2}{18}+\frac{1}{18}a_3^2\right.$$

$$\left.+\frac{1}{2}a_4^2+\frac{1}{2}a_5^2\right\}+\text{part due from } \tfrac{1}{4}m\lambda^2 b_{n-1}^2.$$

On substitution from (18) and (33) we have

$$\bar{P}=-\frac{ma^2u^4}{9\lambda^6}(n^2-3n+2)+\frac{ma^2}{8\lambda^2}\left\{a_1^2+\ldots+\frac{1}{2}a_5^2\right\}$$

$$+\frac{1}{4}ma^2 f(n-1) \quad \ldots \quad (35)$$

Kinetic energy, or K, due to ξ

$$=m\dot{\xi}(\dot{x}_0+\tfrac{1}{2}\dot{\xi})$$

$$=-\frac{ma}{2\lambda}\left[\frac{2}{3}a_2 \sin 2\lambda t-\frac{2}{3}a_3\cos 2\lambda t-a_4\sin \lambda t+a_5\cos \lambda t\right]$$

$$\times\left[\dot{x}_0-\frac{a}{4\lambda}\left\{\frac{2}{3}a_2\sin 2\lambda t-\cdots+a_5\cos \lambda t\right\}\right]$$

$$+\text{Part due from } \tfrac{1}{2}m(b_{n-1}\lambda \sin \lambda t+u\cos \lambda t)^2 \quad \ldots \quad (36)$$

$$\therefore \bar{K}=-\frac{ma}{2}\left[-\frac{a_4 B_{n-1}}{2}+\frac{a_5 u}{2\lambda}\right]+\frac{ma^2}{8\lambda^2}\left\{\frac{2}{9}a_2^2+\frac{2}{9}a_3^2+\frac{1}{2}a_4^2+\frac{1}{2}a_5^2\right\}$$

$$+\text{ part due from } \frac{1}{4}m\lambda^2 b_{n-1}^2.$$

$$=-\frac{ma^2u^4}{9\lambda^6}(n^2-3n+2)+\frac{ma^2}{8\lambda^2}\left\{\frac{2}{9}\ a_2^2+\ldots+\tfrac{1}{2}a_5^2\right\}+\frac{1}{4}ma^2f(n-1).\quad(37)$$

as in (35). Since $\overline{\xi}$ is $-ve$, $\dot{\overline{\xi}}$ is zero and $\overline{\text{P}}$ and $\overline{\text{K}}$ are $+ve$,

$$\therefore \text{E}_1^-=-\overline{\text{P}+\text{K}}$$

$$=-\frac{ma^2}{8\lambda^2}\left\{a_1^2-\frac{1}{6}a_2^2-\frac{1}{6}a_3^2\right\} \qquad \ldots (38)$$

As there is the factor a^2 in the above expression we put therein

$$b_{n-1}=\text{B}_{n-1};$$

and from (15) and (12a) we get

$$\text{E}_2=-\frac{ma^2}{8\lambda^2}\left\{n^2-\frac{1}{6}(n-2)^2-\frac{2}{3}(n-1)\right\}\frac{u^4}{\lambda^4}$$

$$=-\frac{15}{144}\frac{ma^2u^4n^2}{\lambda^6}. \qquad \ldots (39)$$

Putting $\frac{1}{2}mu^2=h\nu$, $\lambda=2\pi\nu$ and $ma=3a_0$, we get

$$\text{E}(n)=\text{E}_0+\text{E}_1$$

$$=nh\nu-\frac{15}{32\nu}\frac{a_0^2h^2n^2}{(2\pi^2\nu m)^3} \qquad \ldots (40)$$

Note: It is necessary to show that $\overline{\text{P}}$ and $\overline{\text{K}}$ are positive. We begin by evaluating $a^2f(n)$ of the equation (33). From (20) and (28) we get

$$\text{R}_{n-1}^3=\text{B}_n^3\left[1+\frac{a}{\lambda^2}\frac{\text{B}_{n-1}^3}{\text{B}_n^2}\right] \text{ and } b_{n-1}^3=\text{B}_{n-1}^3\left[1+\frac{a}{\lambda^2}\text{B}_{n-1}\right].$$

Substituting in (32) and simplifying we obtain

$$\lambda^2b_n^2=\lambda^2b_{n-1}^2+u^2+\frac{2a}{3}(\text{B}_n^3-\text{B}_{n-1}^3)$$

$$+\frac{a^2}{9\lambda^2}\left[4\text{B}_{n-1}^3\text{B}_n+\frac{u^4}{\lambda^4}\left\{n^2-6\ (n-1)^2+\frac{(n-1)^3}{n}\right.\right.$$

$$+4n+4(n-1)+\frac{(n-1)^2}{n}\Big\}.$$

as $$B_n^2=\frac{nu^2}{\lambda^2}.$$ Now $B_{n-1}^3 B_n$ lies betwe

that is,

$$(n-1)^2\frac{u^4}{\lambda^4}\leq B_{n-1}^3\, B_n\leq n^2.\frac{u^4}{\lambda^4}$$

Thus we obtain that

$\lambda^2 b_n^2$ lies between $\lambda^2 b_{n-1}^2+u^2+\frac{2a}{3}(B_n^3\ -B_{n-1}^3)+$

and $$\lambda^2 b_{n-1}^2+u^2+\frac{2a}{3}(B_n^3\ -B_{n-1}^3)+\frac{a^2u^4}{\lambda^6}(2n-$$

So we can write

$$\lambda^2 b_n^2=\lambda^2 b_{n-1}^2+u^2+\frac{2a}{3}(B_n^3\ -B_{n-1}^3)+\frac{\beta a^2}{9\lambda}$$

where $5\leq\beta\leq9$. Similarly

$$\lambda^2 b_{n-1}^2=\lambda^2 b_{n-2}^2+u^2+\frac{2a}{3}(B_{n-1}^3-B_{n-2}^3)+\frac{\beta a^2 u}{9\lambda^6}$$

and so on. Lastly

$$\lambda^2 b_1^2\ =u^2+\frac{2a}{3}(B_1^3)+\frac{\beta a^2u^4}{9\lambda^6}.$$

$\therefore$ adding up and removing common terms we g

$$\lambda^2 b_n^2=nu^2+\frac{2a}{3}B_n^3+\frac{pa^2u^4}{9\lambda^6}n^2$$

where $5\leq p\leq9$. Therefore

$$f(n)=\frac{pu^4}{9\lambda^6}n^2.$$

With this value

$$\overline{P} = \frac{ma^2u^4}{144\lambda^6}[(7+4p)n^2+8n(9-p)+4(p-11)] \quad \dots \quad (46)$$

and

$$\overline{K} = \frac{ma^2u^4}{144\lambda^6}[(4p-8)n^2+8n(9-p)+4(p-11)] \quad \dots \quad (47)$$

which can easily be seen to be positive.

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